The second equality may be rewritten as

$$\phi^{2k} > r\,. \tag{8.10}$$

From the first inequality of (8.9) and (8.10) it follows that

$$\frac{\Delta}{5}\,\phi^{2h+1}\phi^{2k} = \frac{\Delta}{5}\,\phi^{2(h+k)+1} > r\,.$$

Hence

$$\frac{\Delta}{5}\,\phi^{2m+1} > r\,.$$

Since

$$r > \frac{1}{\sin\frac{\pi}{r+2}} \quad \text{for } r \geq 2\,,$$

$m \geq p$ for r sufficiently large and the proof is concluded. □

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