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A DYNAMICAL SYSTEMS APPROACH TO BIRKHOFF'S THEOREM

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ABSTRACT. We present a new proof of Birkhoff's classical theorem that an embedded homotopically nontrivial circle, which is invariant under a monotone twist map on $\mathbf{S}^1 \times \mathbf{R}$, must be the graph of a Lipschitz function.

1. INTRODUCTION

Consider the two-dimensional cylinder $\mathbf{S}^1 \times \mathbf{R} \cong \mathbf{R}/\mathbf{Z} \times \mathbf{R}$, respectively its universal cover $\mathbf{R}^2$ with coordinates x, y. A diffeomorphism $\phi \colon \mathbf{S}^1 \times \mathbf{R} \to \mathbf{S}^1 \times \mathbf{R}$ is called a monotone twist mapping if it is area-preserving and satisfies the monotone twist condition $\partial(\pi_x \circ \phi)/\partial y \neq 0$, where π_x denotes the projection onto the first coordinate. This means, in particular, that (pre-)images of verticals under any lift of ϕ are graphs over the x-axis.

The twist condition is not as artificial as it might seem. Monotone twist mappings appear in a variety of situations, often unexpected and only discovered by clever coordinate choices. In the following, we give a few examples. The reader may consult [LCa, MF, Mo1, Mo2] for more detailed information and further references.

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