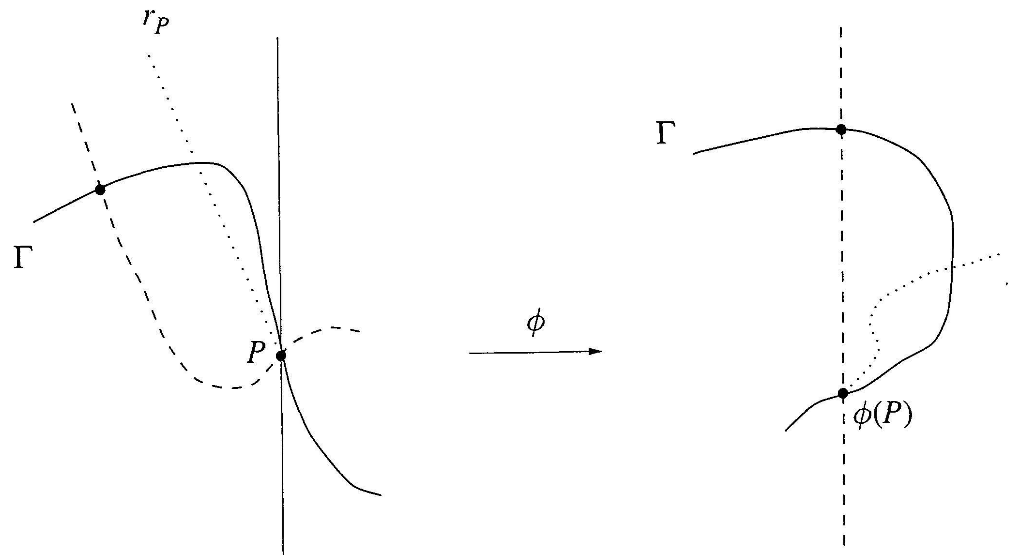

FIGURE 5
Why Γ must be a Lipschitz graph

III. THE SECOND CASE

Finally, the same remark can be applied in the second of the two cases from Lemma 1 where Γ contains a whole vertical interval. For we may take P to be the midpoint of that interval and apply ϕ once – the vertical through $\phi(P)$ will intersect Γ in two isolated points D_0 and E_0, and we are back in the first situation we already dealt with.

The proof of the theorem is complete.

3. CONCLUDING REMARKS

For the sake of clarity, we did not prove the most general result that can be obtained by our method. Here we just indicate possible generalizations.

First of all, our proof does not require the monotone twist condition but only a sort of "cone condition on Γ". Namely, what we really need is the requirement that all (pre-)images of verticals lie outside certain cones centred at points on Γ; we do not use the much more restrictive fact that they are graphs. (This subtle point might be the reason why we have not succeeded in proving a well-known generalization of Birkhoff's Theorem to boundaries of invariant annuli [Fa, He, KH] by our method.)

Secondly, Birkhoff's Theorem also holds true for invariant curves of products $\phi_N \circ \cdots \circ \phi_1$ of monotone twist mappings *of the same sign*. In general, such products are not monotone twist mappings anymore. This generalization follows immediately by our method if, even more generally, each ϕ_n satisfies the same "cone condition" on $(\phi_{n-1} \circ \cdots \circ \phi_1)(\Gamma)$. For every single ϕ_n presses more area into a fold, and $\sup_{n \geq 0} |\Omega_n| < \infty$ because Γ is mapped onto itself again after N steps, instead of one. A proof along the traditional lines was given by Mather only a couple of years ago [Ma3, Appendix].

Finally, we did not really need that ϕ is a diffeomorphism. Everything can also be formulated and proved for homeomorphisms that preserve Lebesgue measure and satisfy the "cone condition".

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