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Artikel: RECENT DEVELOPMENTS ON SERRE'S MULTIPLICITY CONJECTURES: GABBER'S PROOF OF THE NONNEGATIVITY CONJECTURE

Kapitel: 4. SOME EXAMPLES

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4. Some examples

We give here three simple examples of the type of bigraded ring which might result from this construction. Each of these examples is obtained by taking a birational map from a regular scheme to $\operatorname{Spec}(R/\mathfrak{q})$, and the last two are simple resolutions of singularities.

We first summarize the construction up to this point. We began with a regular local ring R and prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. We then took a regular subscheme Z' of $\operatorname{Proj}(R[X_0, \dots, X_n])$ which was generically finite over $\operatorname{Spec}(R/\mathfrak{q})$. The next step was to replace $R[X_0, \dots, X_n]$ with the associated graded ring of I tensored with $R/\mathfrak{m} = k$. The sheaf $\mathcal{O}_{Y'}$ defined by $B = R/\mathfrak{p}[X_0, \dots, X_n]$ was then replaced with the sheaf $\mathcal{M}$ defined by $gr_I(B)$, again tensored with k. The assumption of regularity implies that I/I^2 is locally free over A/I; denote its rank r. Then the dimension of $\mathcal{M}$ is at most r, and it is equal to r if and only if we had $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. We note that the fiber Z'_s of Z' over the maximal ideal of R has dimension at most $\dim(R/\mathfrak{q}) - 1$, but apart from that we do not know much about it. It is the projective scheme defined by the graded ring $(A/I) \otimes_R k$, which is the part of degree zero in the grading in the second component of the bigraded ring we are considering.

For the first example, let R have dimension four, let t, u, v, w be a regular system of parameters, and define the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ by letting $\mathfrak{p} = (t, u)$ and $\mathfrak{q} = (v, w)$. In this case, $\operatorname{Spec}(R/\mathfrak{q})$ is already regular, and we can simply take the projective scheme $\operatorname{Proj}(R[X]) = \operatorname{Spec}(R)$.

For a slightly more complicated example, consider the subscheme of the projective space $\operatorname{Proj}(R[X, Y])$ defined by the ideal I generated by v, w, and $Xu - Yt$. Then Z' is the blow up of $R/\mathfrak{q}$ at the point defined by the maximal ideal, and the fiber over s is projective space of dimension 1. One could define similar examples in higher dimension.

For a third example, let R have dimension 2, and let I be generated by $Xu - Yt, Zu - Xt, X^2 - YZ$. The projective space P has dimension 2, and the fiber over the maximal ideal has codimension one in $\operatorname{Proj}(k[X, Y, Z])$ and thus has dimension one. The sheaf defined by I/I^2 has rank 2, but I is minimally generated by three elements.

In the above examples it was not really necessary to reduce to projective space since the original quotients $R/\mathfrak{q}$ were regular. We next consider an example where the original scheme is not regular. Let $\mathfrak{m}$ be minimally generated by t, u, and let $\mathfrak{q}$ be the principal ideal generated by $t^2 - u^3$.

We can resolve the singularity by letting Z' be defined by the ideal in $R[X,Y]$ generated by t^2-u^3, $uX-tY$, X^2-uY^2. The fiber Z'_s in this case is $\mathrm{Proj}(k[X,Y]/(X^2))$.

Finally, we consider the case where $\mathfrak{q}$ is the determinantal ideal in R of dimension 4 generated by $wu-t^2$, $wv-tu$, and $tv-u^2$. In this case the resolution can be found by taking the ideal I in $R[X,Y,Z,W]$ generated by the following elements:

$$Z^2-YW, YZ-XW, Y^2-XZ, uW-vZ, uZ-vY, uY-vX, u^2-tv,$$
$$tW-vY, tZ-vX, tY-uX, tu-wv, t^2-wu, wW-vX, wZ-uX, wY-tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ for sheaves $\mathcal{M}$ defined as above by certain prime ideals $\mathfrak{p}$ of R.

5. Hilbert polynomials of bigraded modules

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let C denote the bigraded ring which we previously denoted $gr_I(A)\otimes_R k$, where $C_{i,j}$ consists of the elements of $(I^j/I^{j+1})\otimes k$ of degree i. Thus in our present notation, $E_s=\mathrm{Proj}(C)$, where the grading on C is that in the first coordinate. Let C_0 denote the subring $\bigoplus_i(C_{i,0})$. Let r be the rank of I/I^2, and let M be a bigraded module defining a sheaf $\mathcal{M}$ on E_s of dimension at most r; we define the dimension of M to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$, which we also denote $\chi(C_0, M)$.

Let

$$0\to F_k\to\cdots\to F_1\to F_0\to C_0\to 0$$

be a complex of bigraded modules which defines a locally free resolution of C_0 over C. For any finitely generated bigraded module N, we let $P_N(m,n)$ by the Hilbert polynomial of N; more precisely, we define P_N to be the polynomial in two variables such that

$$P_N(m,n)=\sum_{i=0}^{n-1}\mathrm{length}(N_{m,i})$$