HOMFLY POLYNOMIAL VIA AN INVARIANT OF COLORED PLANE GRAPHS

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HOMFLY POLYNOMIAL VIA AN INVARIANT OF COLORED PLANE GRAPHS

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ABSTRACT. After the first discovery of the quantum invariant associated with $SU(2)$ by V.F.R. Jones [3], the invariants associated with $SU(n)$ were found by several authors [1]. It was first proved by V.G. Turaev [16] that all these come from so-called "quantum groups", especially from their R-matrices corresponding to the vector representations. There also exist various quantum invariants corresponding to other representations (see for example [7], [14], [11]).

The aim of this paper is to give a graphical way to define $SU(n)$ quantum invariants for links. To do this we first construct an invariant of colored, oriented, trivalent, plane graphs for each n (≥ 2). Then we show that the $SU(n)$ polynomial invariant corresponding to the vector representation (HOMFLY polynomial) can be defined by using our graph invariant.

We can also show that our invariant defines the $SU(n)$ polynomial invariant corresponding to the anti-symmetric tensors of the vector representation.

We note that our graph invariant for $SU(3)$ was first introduced by G. Kuperberg in [8]. The second and the third authors used it in [12] to construct magic elements and defined the quantum $SU(3)$ invariants for 3-manifolds. Now [12] and the present paper together give an elementary and self-contained proof of the existence of magic elements for $SU(3)$ and so that of the quantum $SU(3)$ invariants of 3-manifolds just like W.B.R. Lickorish did for $SU(2)$ in [9] using the Kauffman bracket [5]. See [17] for a similar approach to $SU(n)$ invariants of 3-manifolds.

We also note that our graph invariant may be obtained (not checked yet) by direct computations of the universal R-matrix. But the advantage of our definition is that it does not require any knowledge of quantum groups nor representation theory. On the contrary we can recover the R-matrix of the quantum group $U_q(\mathfrak{sl}(n, \mathbf{C}))$ corresponding at least to the vector representation.

This work was inspired after conversation with M. Kosuda and J. Murakami, to whom the authors express their gratitude.

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The second named author was in the Department of Mathematical Sciences, University of Tokyo.

§1. Invariants of graphs

In this section we define an invariant of colored, oriented, trivalent, plane graphs.

Fix an integer $n \geq 2$ throughout this paper and put

$$\mathcal{N} = \{-(n-1)/2, -(n-1)/2+1, \ldots, (n-1)/2\}.$$

For disjoint subsets A_1 and A_2 of $\mathcal{N}$ we put

$$\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\}.$$

Let G be an oriented, trivalent, plane graph with "color" or "flow" on each of its edges. Here a *flow* f is a map from the edge set to positive integers less than or equal to n such that for every vertex v in G the sum of its values on the edges coming into v is equal to that on the edges going out from v (see Figure 1.1). So we may say that G is a network with infinite capacity without source or sink. We also note that at each vertex two edges are "in" and one edge is "out", or two edges are "out" and one edge is "in". We call these two in- or out-edges the *legs* and one out- or in-edge the *head* of the vertex.

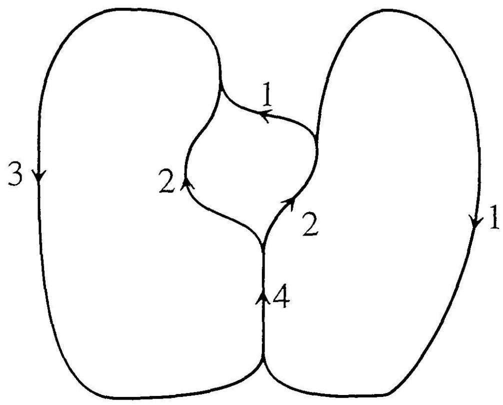


FIGURE 1.1
A graph with flow

A state σ is an assignment of a subset A of $\mathcal{N}$ to each edge e such that $\#(A) = f(e)$ and, moreover, at each vertex the union of subsets assigned to its legs coincides with that assigned to its head, where $\#(A)$ is the number of elements in A (see Figure 1.2). We denote by $\sigma(e)$ the subset of $\mathcal{N}$ assigned to an edge e.

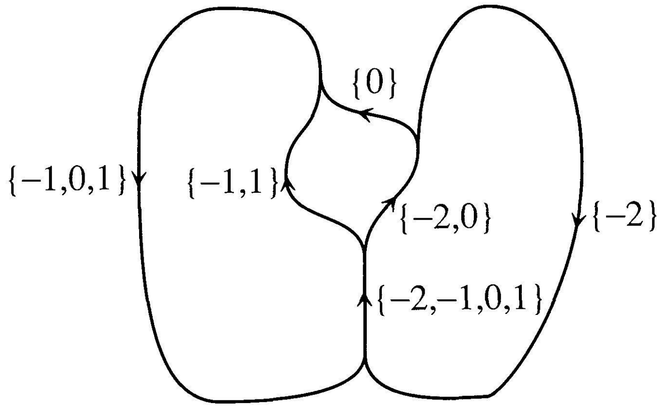


FIGURE 1.2
A state of the graph with flow in Figure 1.1

Given a state σ, we define the *weight* $\mathrm{wt}(v;\sigma)$ of a vertex v to be

$$q^{f(e_1)f(e_2)/4-\pi(\sigma(e_1),\sigma(e_2))/2},$$

where q is an indeterminate, and e_1 and e_2 are left and right legs respectively with respect to the orientation of G (Figure 1.3).

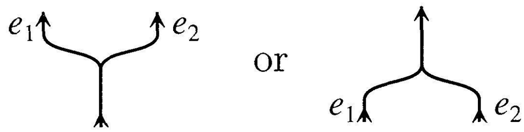


FIGURE 1.3

If we replace every edge e with $f(e)$ copies of parallel edges, assign each copy an element of the subset determined by σ, and connect at every vertex each pair of edges with the same element, we have a union of simple closed curves each of which equipped with the same element of $\mathcal{N}$ (Figure 1.4).

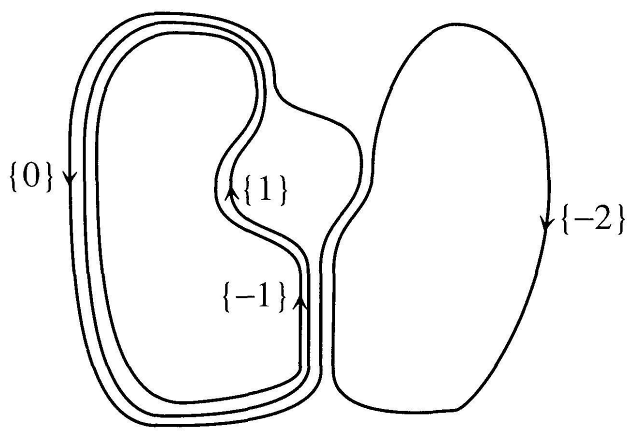


FIGURE 1.4
Simple closed curves defined by Figure 1.2

Then we define the *rotation number* $\mathrm{rot}(\sigma)$ to be

$$\sum_C \sigma(C)\,\mathrm{rot}(C)$$

where the sum is over all simple closed curves C equipped with $\sigma(C) \in \mathcal{N}$ and $\mathrm{rot}(C)$ is the rotation number of C (i.e., 1 if C is counter-clockwise and -1 otherwise). For example $\mathrm{rot}(\sigma) = 2$ for the state described in Figure 1.2 (see Figure 1.4).

Now we define $\langle G \rangle_n$ as follows.

$$\langle G \rangle_n = \sum_{\sigma:\text{state}} \Big\{ \prod_{v:\text{vertex}} \mathrm{wt}(v;\sigma) \Big\} q^{\mathrm{rot}(\sigma)} .$$

We define $\langle \text{empty graph} \rangle_n = 1$. It is clear that this is invariant under ambient isotopy of $\mathbf{R}^2$. Note that our invariant can be regarded as a colored graph invariant introduced by N. Yu. Reshetikhin and V. G. Turaev in [14] replacing each vertex by a "coupon". The coupon with two legs in would correspond to a projection $V_i \otimes V_j \to V_{i+j}$ and that with two legs out to an inclusion $V_{i+j} \to V_i \otimes V_j$, where V_i is the irreducible representation of $SU(n)$ corresponding to the i-fold anti-symmetric tensor of the vector representation.

§2. Local properties of $\langle G \rangle_n$

We will describe some local properties of $\langle G \rangle_n$. In what follows diagrams indicated in each equality are identical outside the angle brackets $\langle \quad \rangle_n$ and each equality also holds if we reverse all the orientations of diagrams in both hand sides. A number near an edge indicates its flow. If a flow in a diagram exceeds n, we disregard the term where the diagram appears.

We put

$$[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}} ,$$

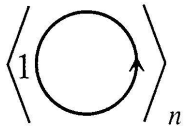
$$[k]! = [1][2] \cdots [k] ,$$

and

$$\begin{bmatrix} i \\ j \end{bmatrix} = \frac{[i]!}{[j]![i-j]!} .$$

In the following equations we mean that if we replace the graph appearing in the left hand side with the one in the right hand side, we obtain the same value.

LEMMA 2.1.

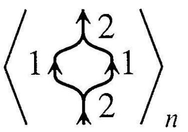
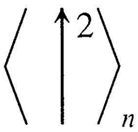
 $= [n]\,\langle\varnothing\rangle_n\,.$

Proof. From the definition the left hand side is equal to

$$\sum_{i=-(n-1)/2}^{(n-1)/2} q^i\,,$$

which is $[n]$, completing the proof. □

LEMMA 2.2.

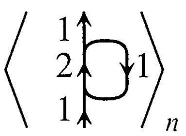
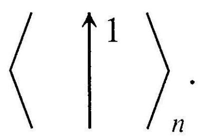
(2.1)  $= [2]$  .

Proof. Consider a state σ of the left hand side. First note that the top-most and the bottom-most edges are equipped with the same subset. So we may put it $\{\alpha, \beta\}$ $(\alpha < \beta)$. Then there are two cases; (i) the left edge is equipped with $\{\alpha\}$ and the right one with $\{\beta\}$ and (ii) the left edge is equipped with $\{\beta\}$ and the right one with $\{\alpha\}$. In the first case the weights of the upper and lower vertex are the same and equal to $q^{1/4}$. In the second case they are also the same and equal to $q^{-1/4}$. Therefore the contribution of the two vertices is

$$q^{1/2} + q^{-1/2} = [2]\,,$$

which is independent of σ and the conclusion follows. □

LEMMA 2.3.

 $= [n-1]$  .

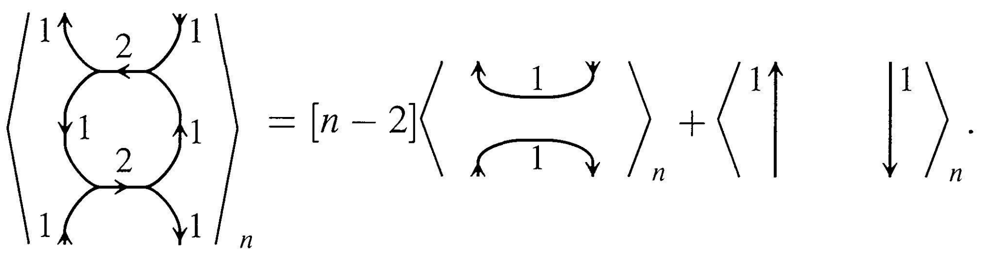
Proof. In this case the top-most and the bottom-most edges of the left hand side are equipped with the same subset for any state σ as the previous lemma. So we put it $\{\alpha\}$. If the right edge is equipped with $\{\beta\}$ $(\beta \neq \alpha)$, then the weights of the two vertices are the same and equal to $q^{\operatorname{sign}(\beta-\alpha)/4}$.

Since its contribution to the rotation number is $-\beta$, the contribution of the left hand side is

$$\sum_{\beta\neq\alpha} q^{\operatorname{sign}(\beta-\alpha)/2-\beta} = \sum_{\beta=-(n-1)/2}^{\alpha-1} q^{-1/2-\beta} + \sum_{\beta=\alpha+1}^{(n-1)/2} q^{1/2-\beta}$$
$$= q^{(n-2)/2} + q^{(n-4)/2} + \cdots + q^{-\alpha+1/2} + q^{-\alpha-1/2} + \cdots + q^{-(n-2)/2} = [n-1]$$

and the proof is complete. □

LEMMA 2.4.



Proof. There are three possibilities:

(1) the two edges at the top are equipped with $\{\alpha\}$ and the two edges at the bottom are equipped with $\{\beta\}$ $(\alpha \neq \beta)$;

(2) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\beta\}$ $(\alpha \neq \beta)$;

and

(3) all the four edges at the corners are equipped with $\{\alpha\}$.

In the first case, the two horizontal edges are equipped with $\{\gamma\}$ $(\alpha \neq \gamma \neq \beta)$. So the contribution of the left hand side is

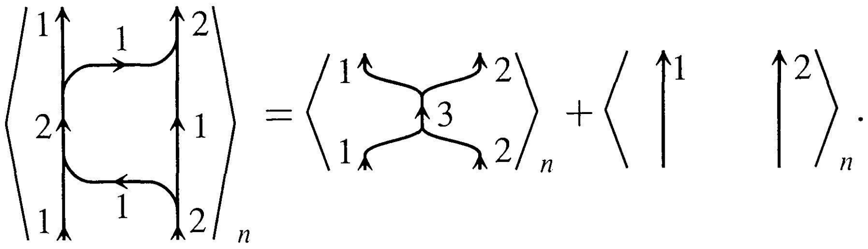
$$\Big(\sum_{\substack{\gamma \\ \alpha\neq\gamma\neq\beta}} q^{\operatorname{sign}(\alpha-\gamma)/2+\operatorname{sign}(\beta-\gamma)/2} q^{\gamma}\Big) q^{-(\alpha+\beta)/2} = [n-2]\, q^{-(\alpha+\beta)/2}$$

which is equal to the contribution of the right hand side.

In the second case, the contribution of the first term of the right hand side is zero. It is easy to see that the left hand side and the second term of the right hand side coincide.

In the third case, the contribution of the left hand side is $[n-2]\, q^{-\alpha} + 1$, which is equal to that of the right hand side. This completes the proof. □

LEMMA 2.5.



Proof. There are three possibilities:

(1) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\beta, \gamma\}$ (α, β and γ are all distinct);

(2) the top left and the bottom left edges are equipped with $\{\alpha\}$ and $\{\beta\}$ respectively and the top right and the bottom right edges are equipped with $\{\beta, \gamma\}$ and $\{\alpha, \gamma\}$ respectively (α, β and γ are all distinct);

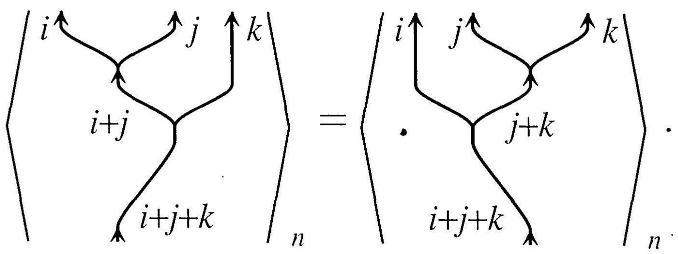
and

(3) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\alpha, \beta\}$ ($\alpha \neq \beta$).

In the first case both the upper and the lower horizontal edges in the left hand side are equipped with $\{\beta\}$ or both of them are equipped with $\{\gamma\}$. So the contribution of the left hand side is $q^{\text{sign}(\beta-\alpha)/2+\text{sign}(\gamma-\beta)/2} + q^{\text{sign}(\gamma-\alpha)/2+\text{sign}(\beta-\gamma)/2}$. On the other hand that of the right hand side is $q^{1-\pi(\{\alpha\},\{\beta,\gamma\})} + 1$. It can be easily checked that these are the same.

The second and the third cases are easily checked and left to the reader. □

LEMMA 2.6.



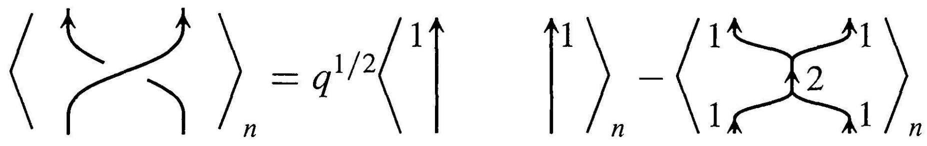
Proof. This follows from the fact that

$$\begin{aligned}
&\#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} + \#\{(a, a_3) \in (A_1 \cup A_2) \times A_3 \mid a > a_3\} \\
&\quad = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} \\
&\qquad + \#\{(a_1, a_3) \in A_1 \times A_3 \mid a_1 > a_3\} + \#\{(a_2, a_3) \in A_2 \times A_3 \mid a_2 > a_3\} \\
&\quad = \#\{(a_1, a) \in A_1 \times (A_2 \cup A_3) \mid a_1 > a\} + \#\{(a_2, a_3) \in A_2 \times A_3 \mid a_2 > a_3\}.
\end{aligned}$$

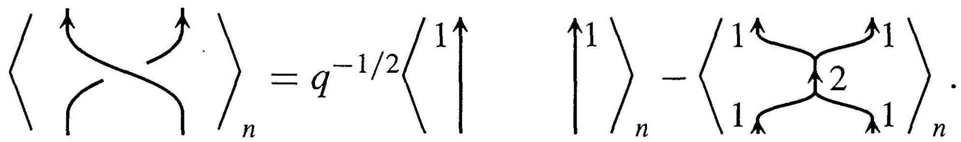
□

§ 3. Invariants of links

In this section, we give a graphical definition of the HOMFLY polynomial of oriented links. For an oriented link diagram D, we define $\langle D \rangle_n$ by the following.



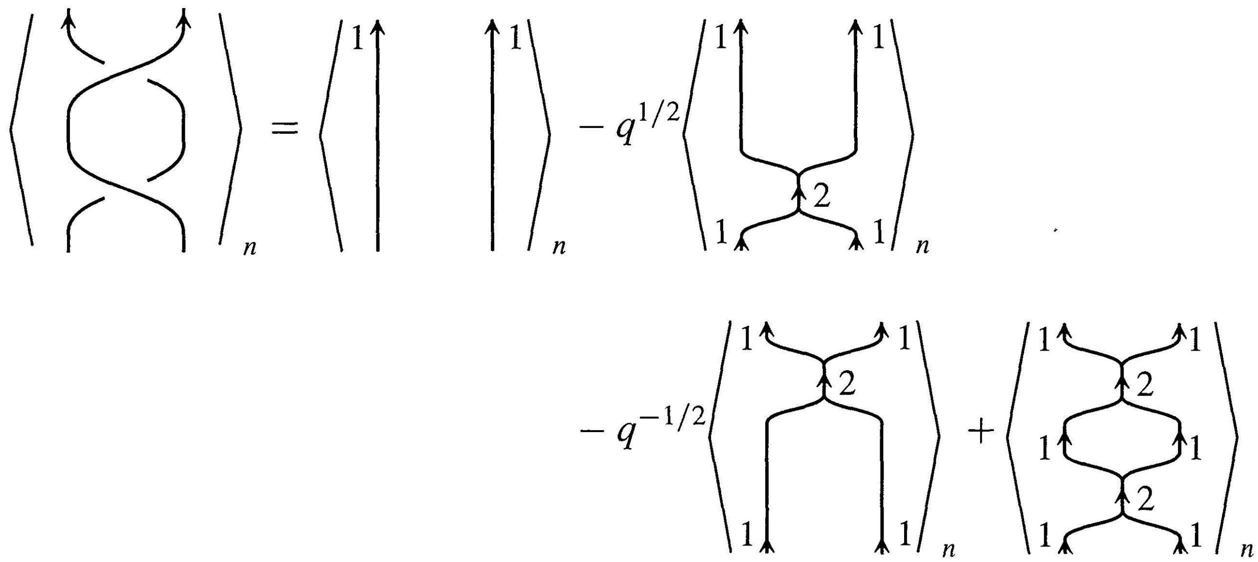
and

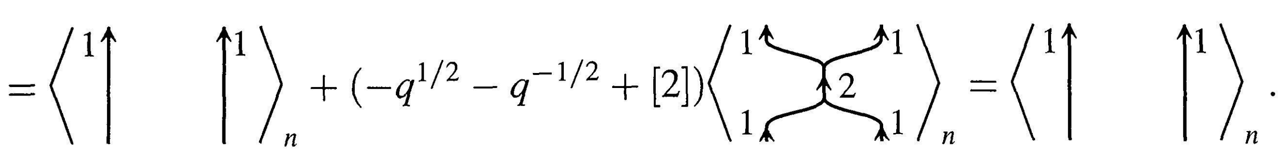


Then we have

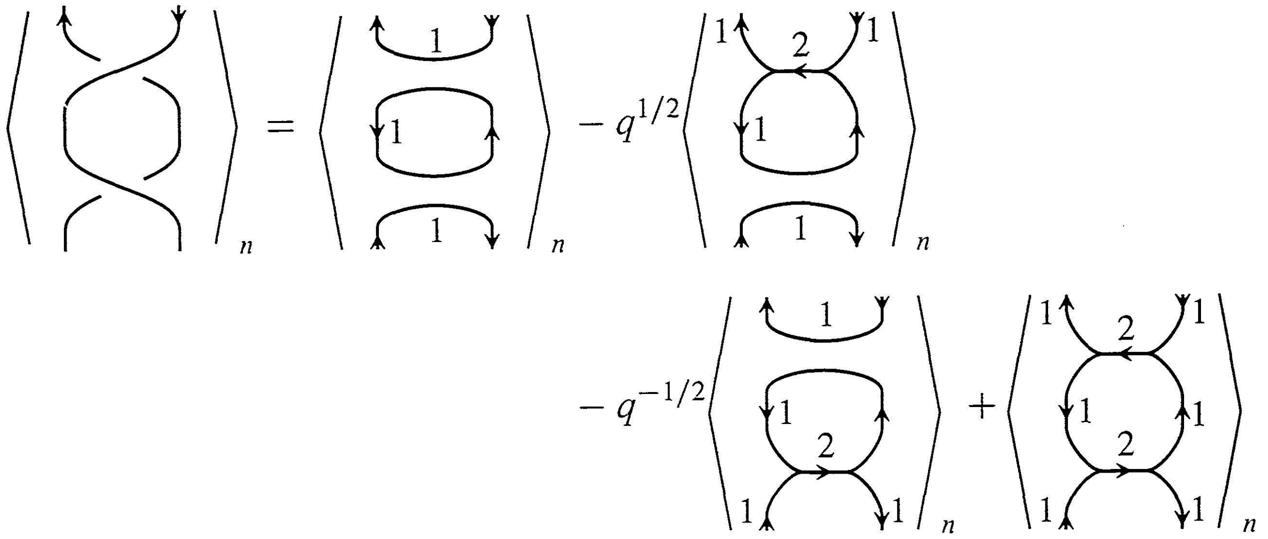
THEOREM 3.1. *$\langle D \rangle_n$ is invariant under the Reidemeister moves II and III.*

Proof. From Lemma 2.2, we have

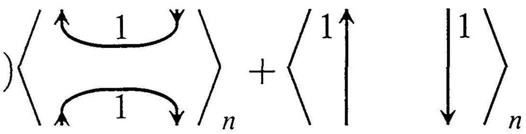


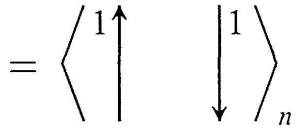


We also have

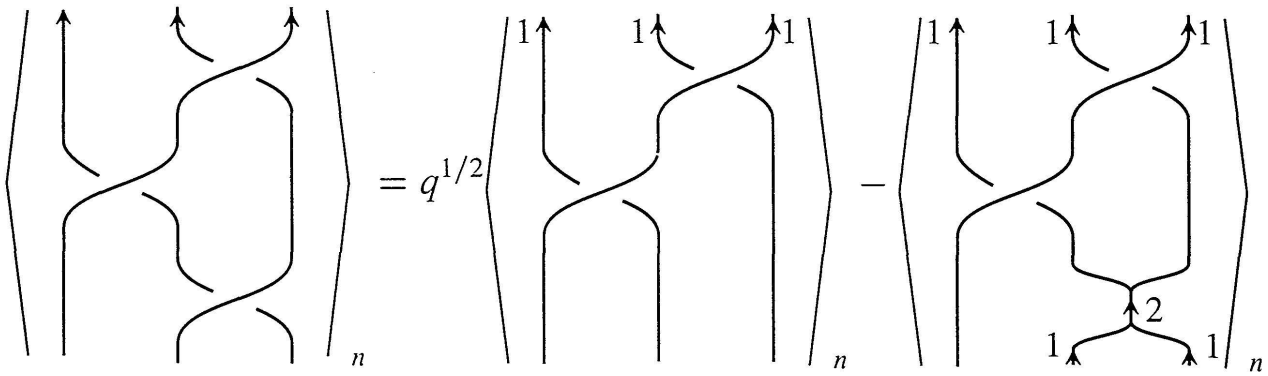


$= ([n] - q^{1/2}[n-1] - q^{-1/2}[n-1] + [n-2])$

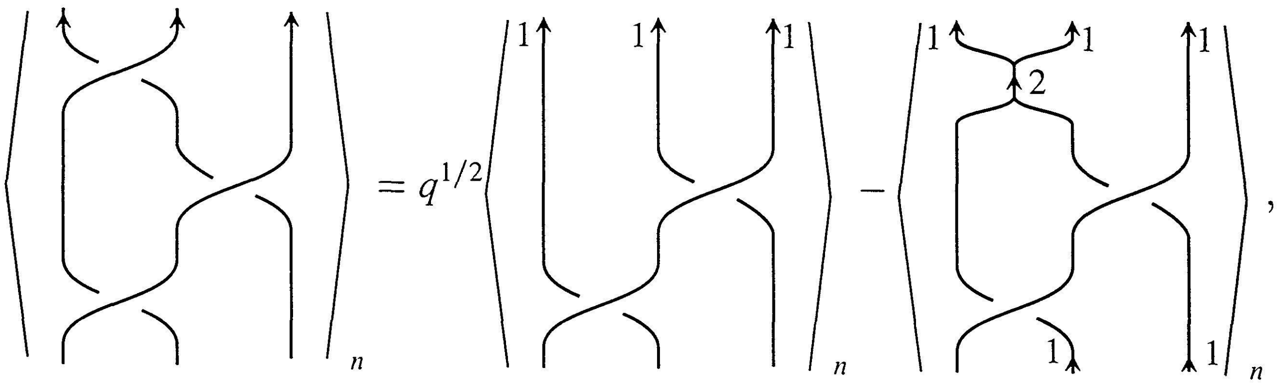




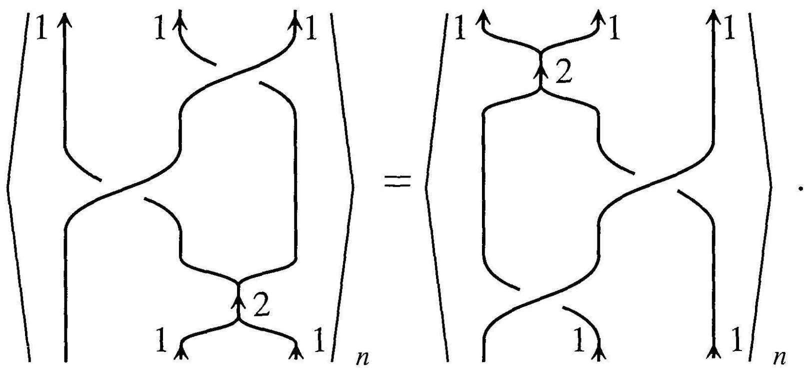
from Lemmas 2.1, 2.3 and 2.4 and so $\langle D \rangle_n$ is invariant under the Reidemeister move II. Next we prove the invariance under the Reidemeister move III. Since we have



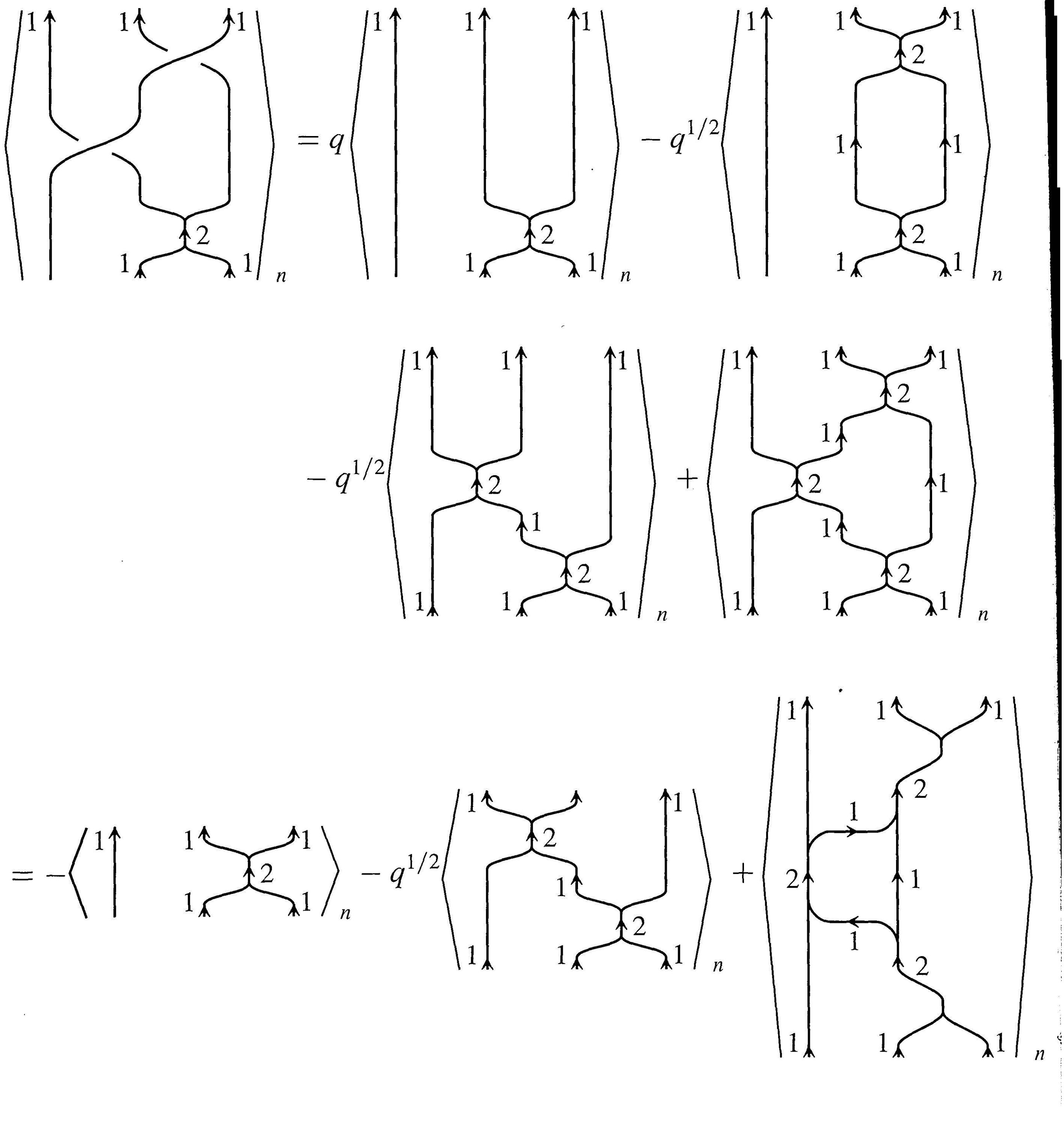
and

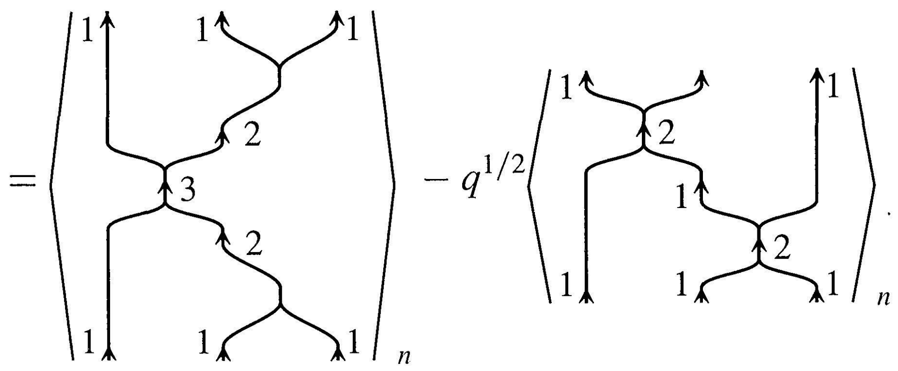


it suffices to show that

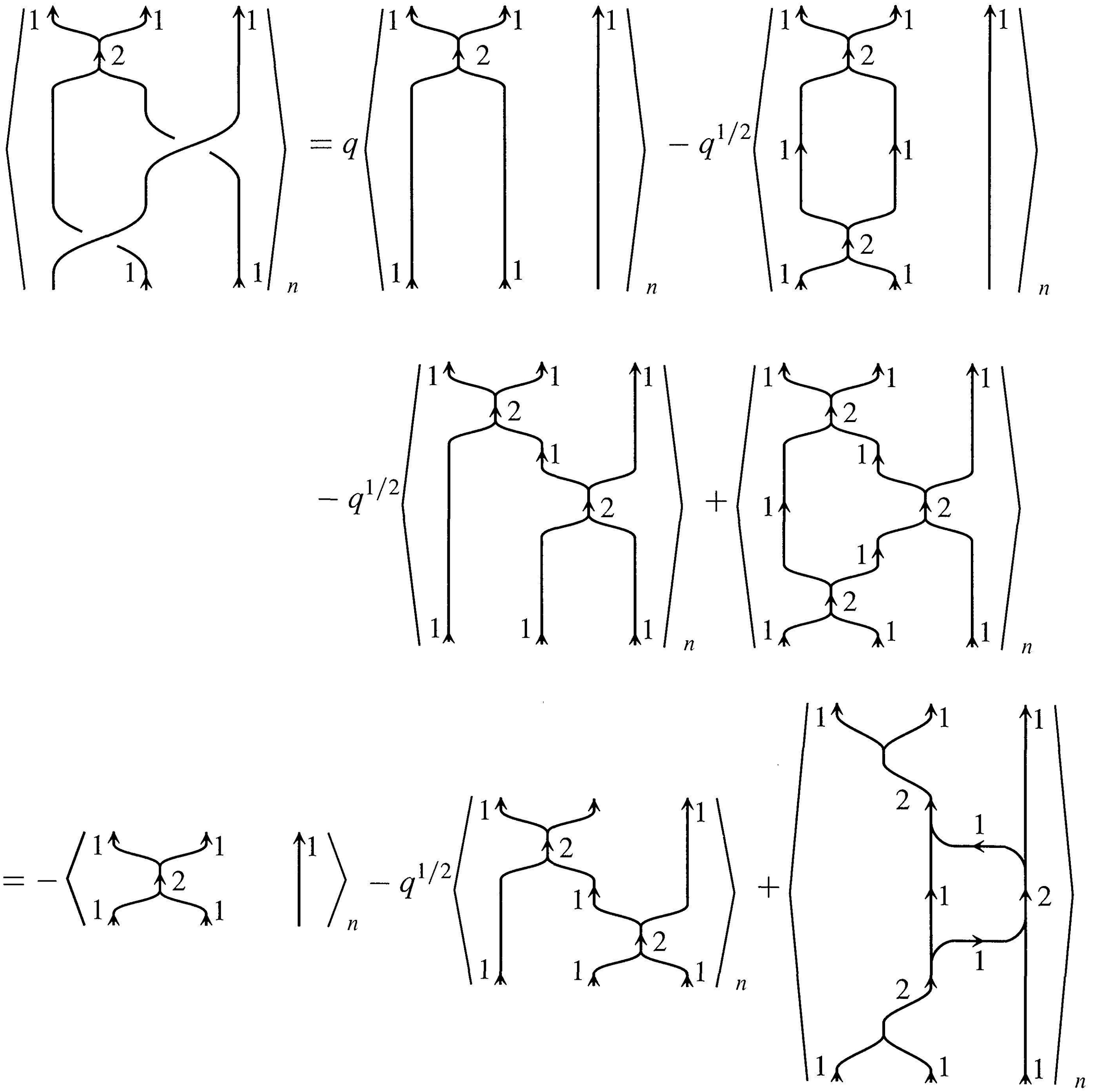


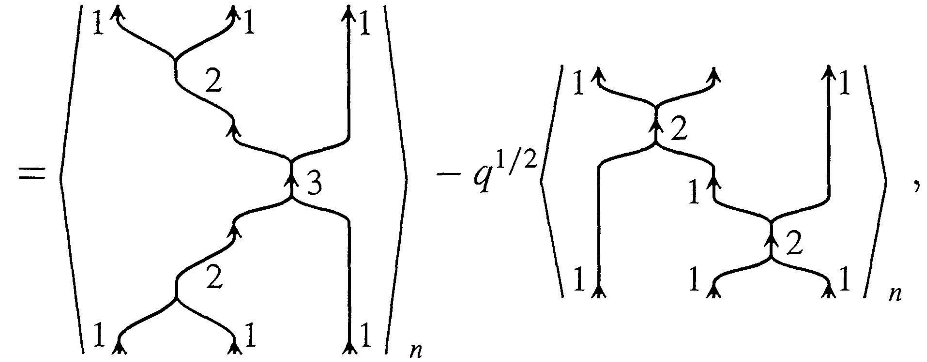
Since





from Lemmas 2.2 and 2.5 and





we have the required formula from Lemma 2.6.

Since the Reidemeister move III with other types of orientations can be obtained from the Reidemeister moves II and III described above (see [12]), the proof is complete. □

If we define $P_n(D) = q^{(n/2)(-\mathrm{w}(D))} \langle D \rangle_n$ with $\mathrm{w}(D)$ the writhe (the algebraic sum of the crossings) of D, then we have

THEOREM 3.2. *$P_n(D)$ is invariant under the Reidemeister moves I, II, and III and satisfies the following skein relation.*

$$q^{n/2}P_n(D_+) - q^{-n/2}P_n(D_-) = (q^{1/2} - q^{-1/2})P_n(D_0),$$

where D_+, D_- and D_0 are identical link diagrams except near a crossing as described in Figure 3.1.

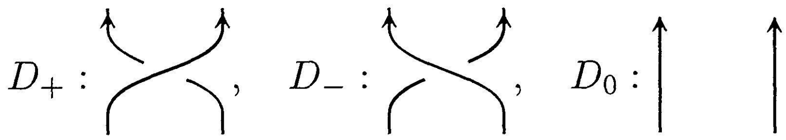
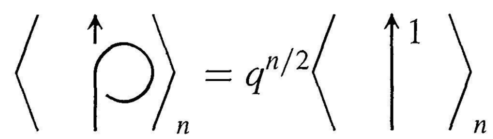


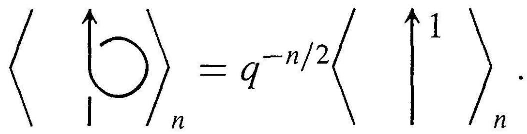
FIGURE 3.1
skein triple

The proof of this theorem follows immediately from the following lemma.

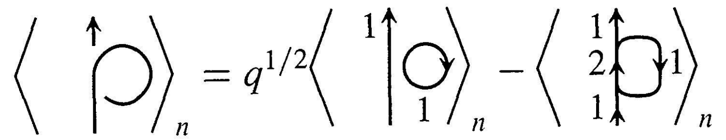
LEMMA 3.3.

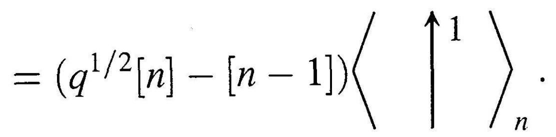


and



Proof. From Lemmas 2.1 and 2.3, we have

(3.1) 

(3.2) 

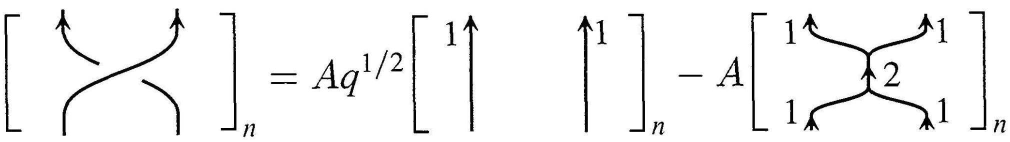
Since

$$q^{1/2}[n] - [n-1] = q^{n/2},$$

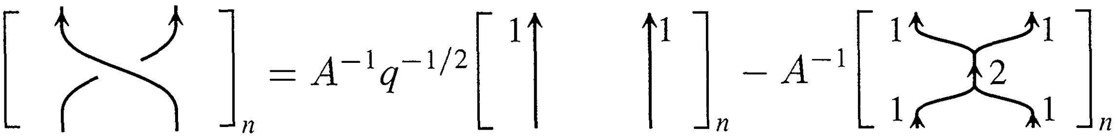
the first equality follows. The second equality follows similarly and the proof is complete. □

Therefore P_n defines a link invariant and so we can put $P_n(L) = P_n(D)$ for the link L presented by D. Then P_n is a version of the HOMFLY polynomial [1], [13].

REMARK 3.4. If we define $[D]_n$ to be



and



with A an indeterminate, we also have a framed link invariant.

When $n = 2$ and $A = q^{-1/4}$, we have a version of the Kauffman bracket naturally defined from representation theory of $U_q(\mathfrak{sl}(2, \mathbf{C}))$ [6, Theorem 4.3].

For $n = 3$ we have G. Kuperberg's recursive formula [8] as follows. Consider an oriented, trivalent, plane graph with flow less than or equal to three. If we reverse the orientations of all the edges with flow two, remove all the edges with flow three, and forget the flow, then we have a trivalent graph without flow such that at every vertex all the edges are in or out. Putting $A = q^{-1/6}$ and replacing q with q^{-1}, we have Kuperberg's formula. (This corresponds to the fact that the two-fold anti-symmetric tensor of the vector representation of $SU(3)$ is isomorphic to its dual.) See also [12].

§4. STATE MODEL OF TURAEV AND JONES

In this section we show that our definition of the HOMFLY polynomial gives the state model due to V. G. Turaev and V. F. R. Jones [16], [4]. Moreover we can recover the R-matrix for the q-deformation of the universal enveloping algebra $U_q(\mathfrak{sl}(n, \mathbf{C}))$ found by M. Jimbo [2].

For readers' convenience, we first review Turaev's state model. Let $R\colon V^{\otimes 2} \to V^{\otimes 2}$ be an isomorphism with V an n-dimensional vector space over $\mathbf{C}$. It is called an R-matrix if it satisfies the so-called Yang-Baxter equation:

$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R).$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V and $R_{i,j}^{k,l}$ be the entry of R with respect to a fixed basis of V, which is the coefficient of $e_k \otimes e_l$ of $R(e_i \otimes e_j)$.

Given a link diagram D, we regard it as a 4-valent graph (a crossing corresponding to a 4-valent vertex) and denote it by $\widetilde{D}$. See Figure 4.1. Here the $+$ ($-$, respectively) sign indicates that the vertex comes from a positive (negative, respectively) crossing.

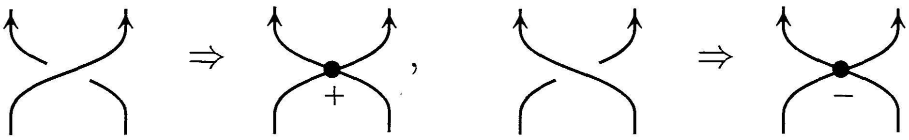


FIGURE 4.1
Change crossings to 4-valent vertices

A state $\widetilde{\sigma}$ of $\widetilde{D}$ is a mapping from the edge set of $\widetilde{D}$ to $\mathcal{N}$. Then we define the weight $\mathrm{wt}_R(\widetilde{v}, \widetilde{\sigma})$ of a vertex $\widetilde{v}$ to be R_{ab}^{cd} if $\widetilde{v}$ comes from a positive crossing and $(R^{-1})_{ab}^{cd}$ if it comes from a negative one, where $a, b, c, d \in \mathcal{N}$ are the values of the four edges adjacent to $\widetilde{v}$ (see Figure 4.2).

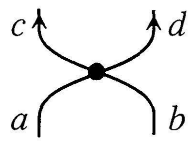


FIGURE 4.2
State around a vertex $\widetilde{v}$

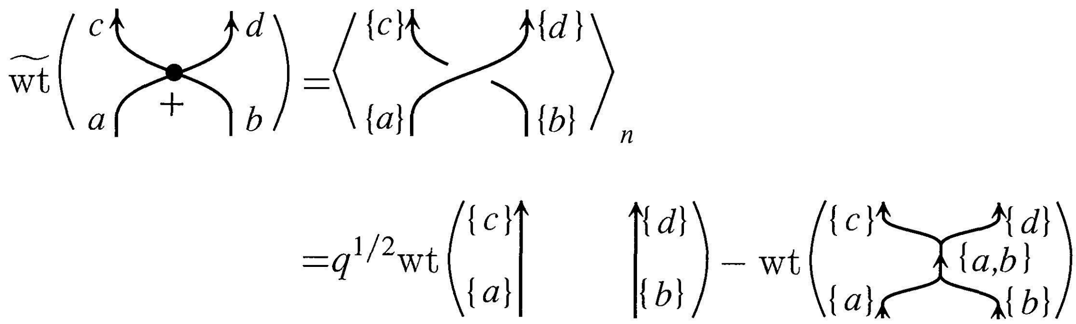
If we can define a suitable "rotation number" $\widetilde{\mathrm{rot}}(\widetilde{\sigma})$ for each state, then the quantity

$$\sum_{\widetilde{\sigma}:\text{state}} \left\{ \prod_{\widetilde{v}:\text{vertex}} \mathrm{wt}_R(\widetilde{v}, \widetilde{\sigma}) \right\} q^{\widetilde{\mathrm{rot}}(\widetilde{\sigma})}$$

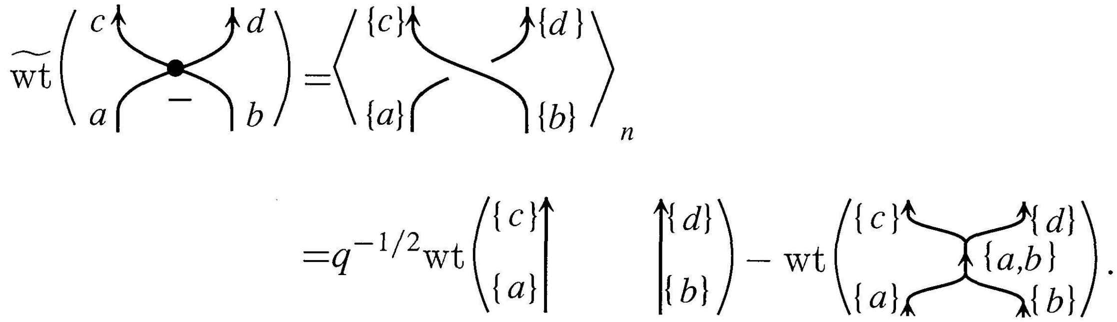
becomes a framed link invariant.

Now we will proceed conversely and define an R-matrix from our framed link invariant $\langle D \rangle_n$.

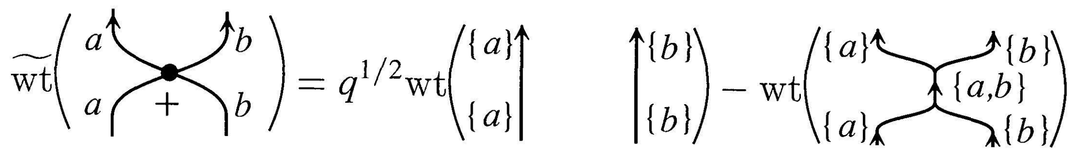
First we define a weight $\widetilde{\mathrm{wt}}(\widetilde{v}, \widetilde{\sigma})$ by using $\langle D \rangle_n$ as follows. We assume that for a state $\widetilde{\sigma}$, $a, b, c, d \in \mathcal{N}$ appear around a vertex $\widetilde{v}$. We define



and



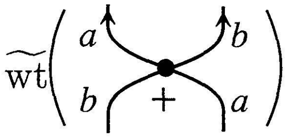
Here the first terms of the right hand sides are zero if $a \neq c$ or $b \neq d$ and the second terms are zero if $\{a, b\} \neq \{c, d\}$. Therefore, for $a, b \in \mathcal{N}$ $(a \neq b)$ we have

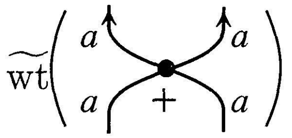


$$= q^{1/2} - q^{\mathrm{sign}(b-a)/2}$$

$$= \begin{cases} q^{1/2} - q^{-1/2} & \text{if } a > b, \\ 0 & \text{if } a < b. \end{cases}$$

Similarly we have



$= -1$ and


$= q^{1/2}$.

We have a similar formula for a negative crossing.

Therefore we see that our graph invariant gives an R-matrix of the form

$$R_{ij}^{kl} = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } i = k > j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(R^{-1})_{ij}^{kl} = \begin{cases} -q^{1/2} + q^{-1/2} & \text{if } i = k < j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{-1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

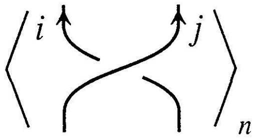
which coincides with $-R(q^{1/2})^{-1}$, where $R(q^{1/2})$ is the R-matrix given in [16], replacing q with $q^{1/2}$.

§5. INVARIANTS CORRESPONDING TO ANTI-SYMETRIC TENSORS

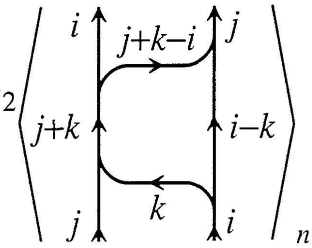
In this section, we will show briefly that our graph invariant gives the quantum link invariant each of its component equipped with an anti-symmetric tensor of the standard n-dimensional representation of $SU(n)$.

Let D be a link diagram each of its component colored with an integer i $(1 \leq i \leq n)$. This corresponds to the i-fold anti-symmetric tensor of the standard representation of $SU(n)$.

Then $\langle D\rangle_n$ is defined by

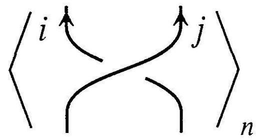


$$= \sum_{k=0}^{i} (-1)^{k+(j+1)i} q^{(i-k)/2}$$

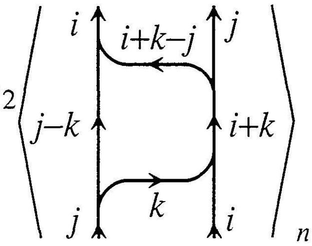


, for $i \le j$

and



$$= \sum_{k=0}^{j} (-1)^{k+(i+1)j} q^{(j-k)/2}$$



, for $i > j$

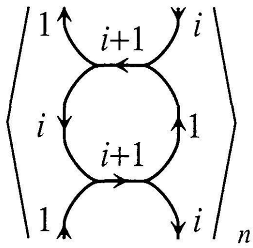
For a negative crossing, replace q with q^{-1}.

Now we will show

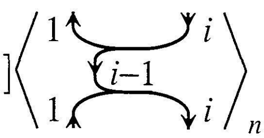
THEOREM 5.1. *The quantity $\langle D\rangle_n$ with D a colored link diagram is invariant under the Reidemeister moves II and III. Thus it is a colored framed link invariant.*

To prove the theorem above, we prepare some lemmas:

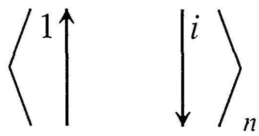
LEMMA 5.2.



$= [n-i-1]$



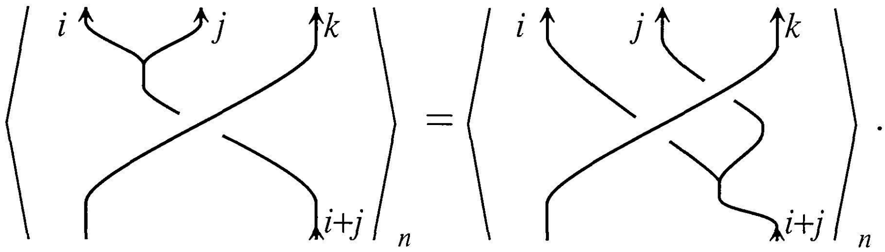
$+$



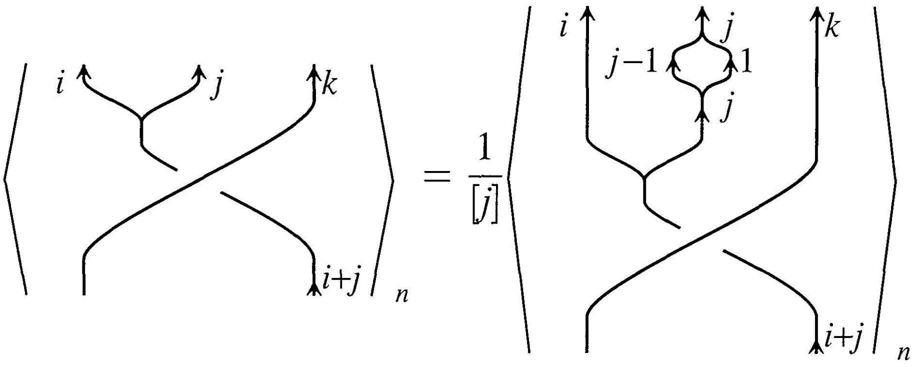
.

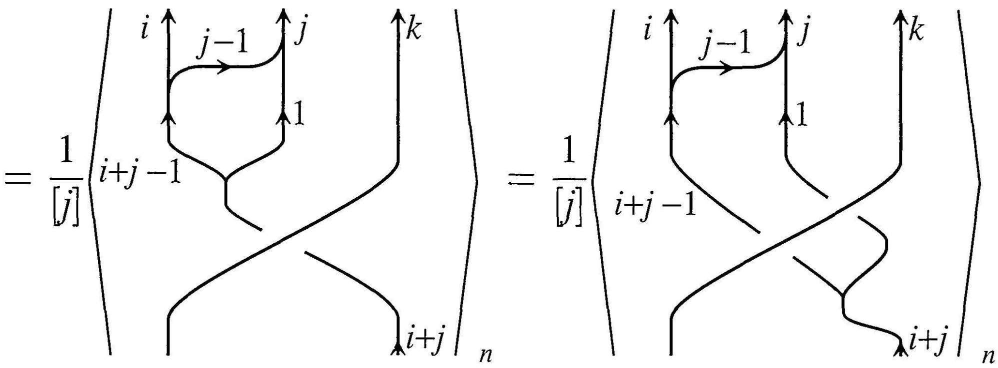
Proof. The proof of this lemma is similar to that of Lemma 2.4 and we leave it to the reader. □

LEMMA 5.3.

(5.1) 

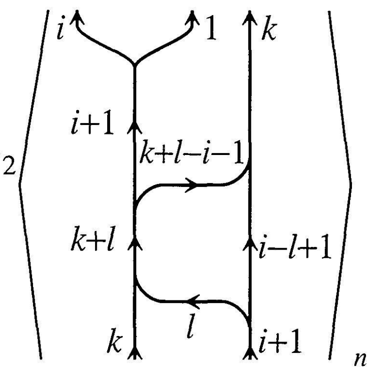
Proof. It suffices to prove the case $i = 1$ or $j = 1$ since we have





and the conclusion follows from the case $i = 1$ or $j = 1$ and induction. Here we use Lemma A.1 in the first equality.

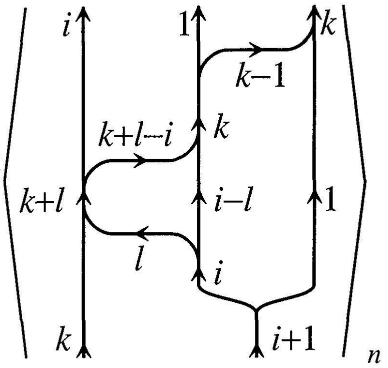
We only prove the case $j = 1$ and $i < k$ since the remaining case is similar. From the definition, the left hand side of (5.1) with $j = 1$ equals

$$\sum_{l=0}^{i+1}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$ 

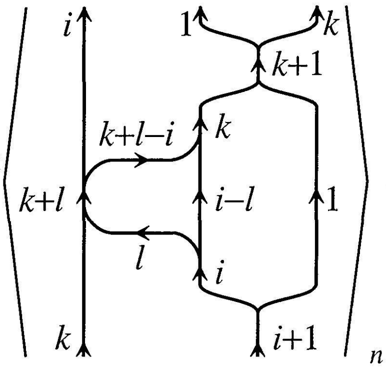
.

The right hand side becomes

$$\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$

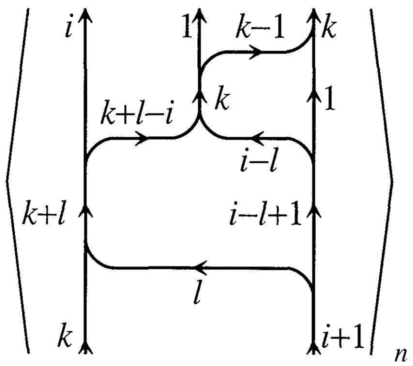


$$+\sum_{l=0}^{i}(-1)^{l+(k+1)i+k}q^{(i-l)/2}$$

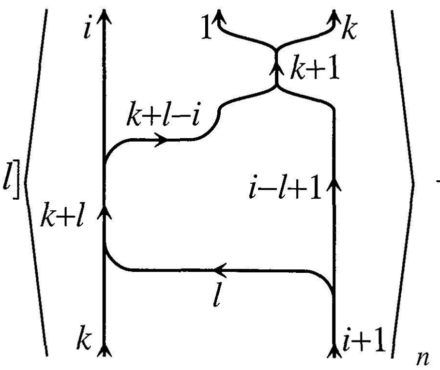


.

Sliding the bar colored with l using Lemma 2.6, the first diagram becomes



$= [i-l]$



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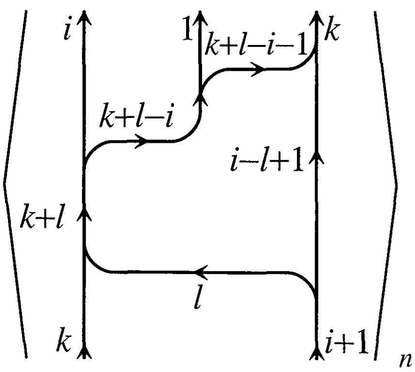
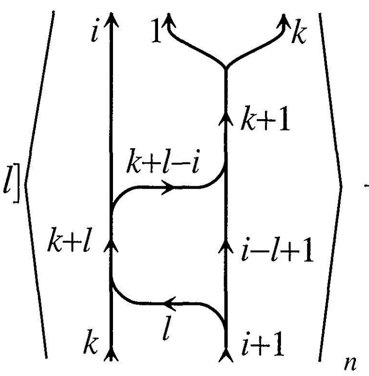


$= [i-l]$



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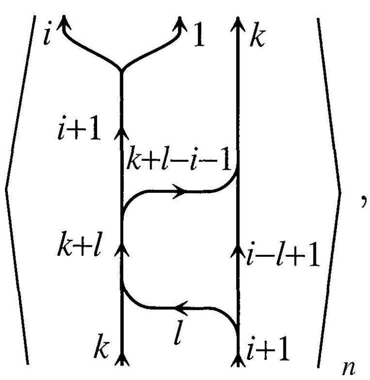
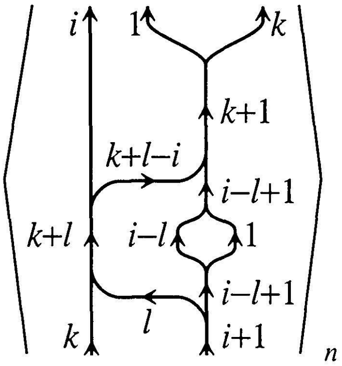


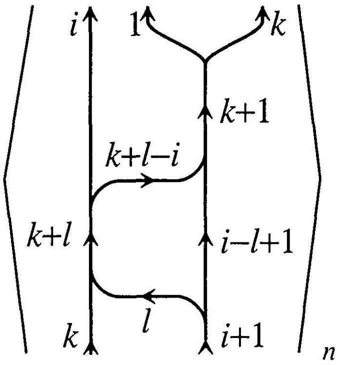
,

where the first equality follows from Lemma A.7 below.

The second diagram turns out to be



$$= [i - l + 1]$$

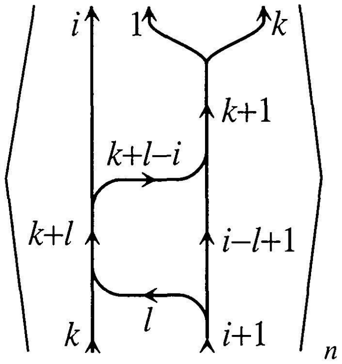


.

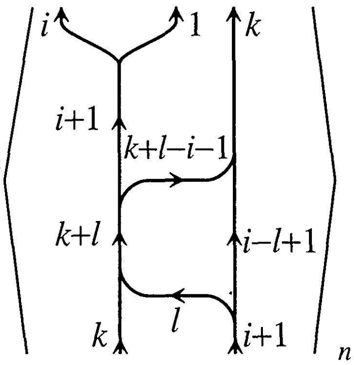
Therefore the right hand side of (5.1) becomes

$$\left\{\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}[i-l]+\sum_{l=0}^{i}(-1)^{l+(k+1)i+k}q^{(i-l)/2}[i-l+1]\right\}$$

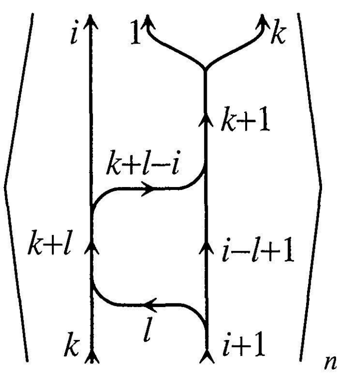
$$\times$$



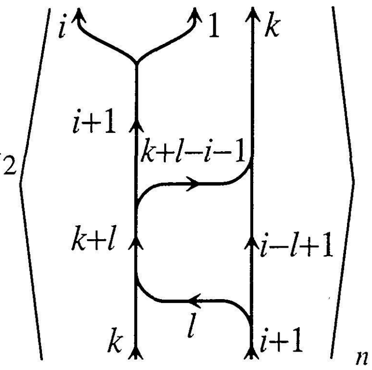
$$+\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$



$$=\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)+1}$$

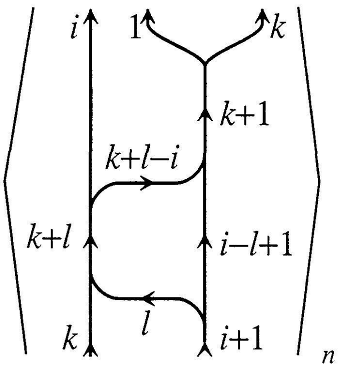


$$+\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)}q^{(i-l+1)/2}$$

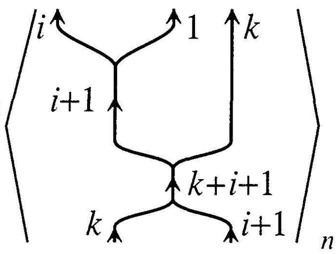


We finally see that the right hand side of (5.1) minus the left hand side equals

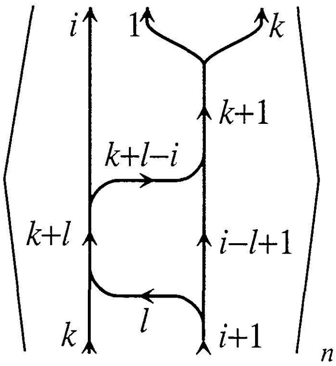
$$\sum_{l=0}^{i}(-1)^{l+(k+1)(i+1)+1}$$



$$-(-1)^{i+1+(k+1)(i+1)}$$



$$=\sum_{l=0}^{i+1}(-1)^{l+(k+1)(i+1)+1}$$

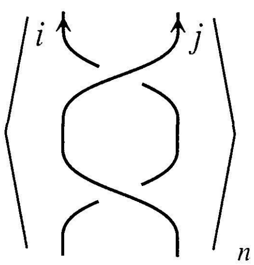


$$=0.$$

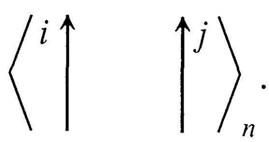
Here the last equality follows from Lemma A.9, completing the proof. □

PROOF OF THEOREM 5.1

The invariance under the Reidemeister move II. We will first show

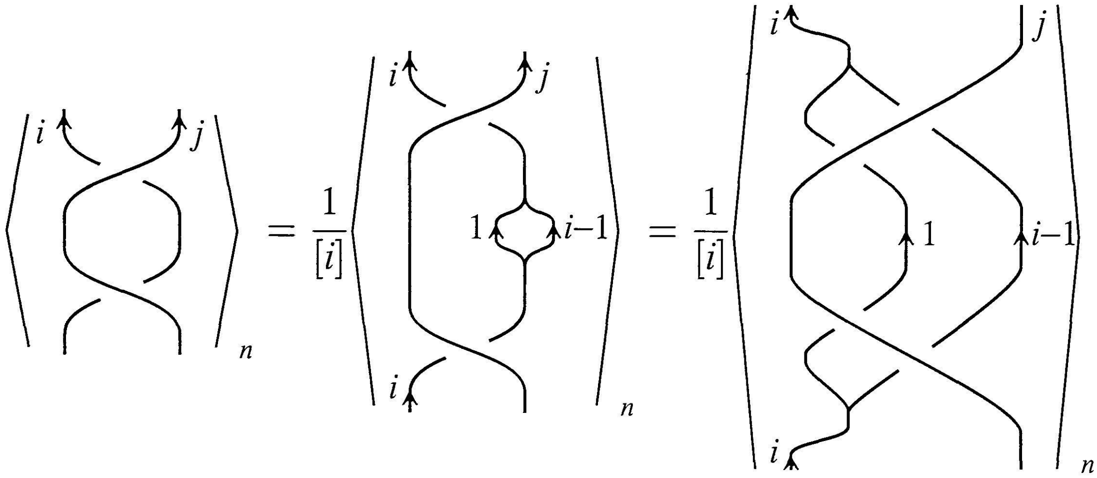


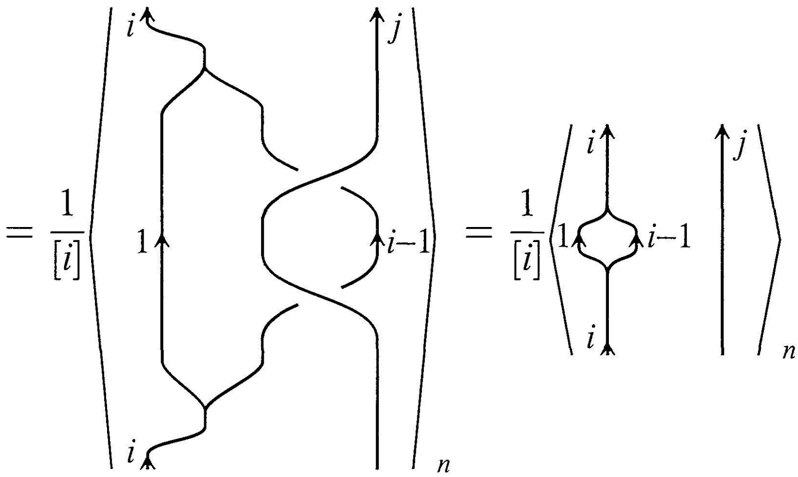
$$=$$

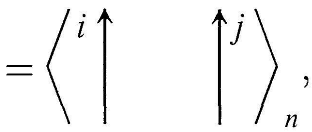


.

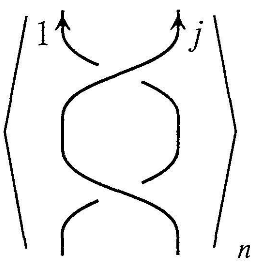
It suffices to show the case $i = 1$ from Lemmas A.1 and 5.3 since

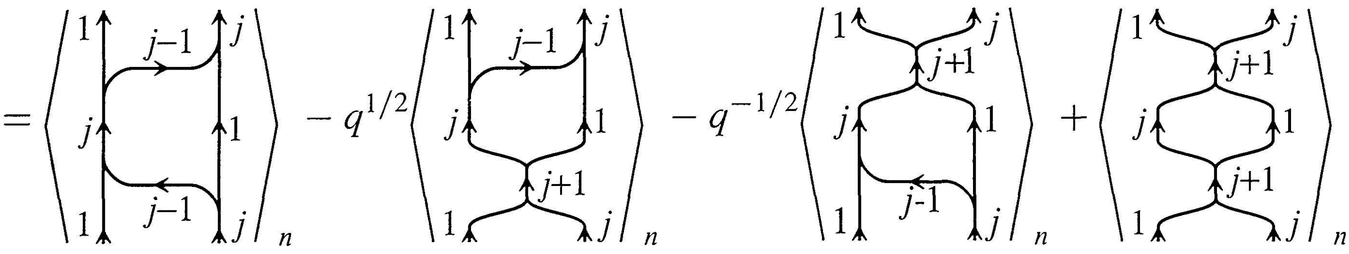


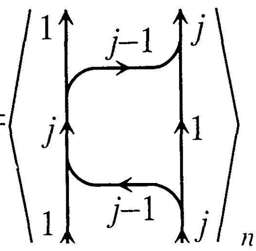
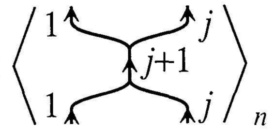


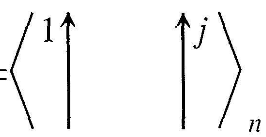
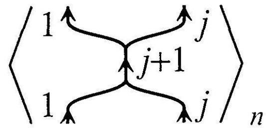


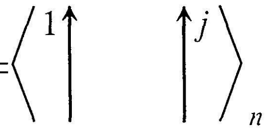
where the second equality follows from Lemma 5.3 and the fourth by induction on i. Now we have





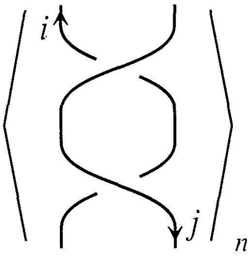
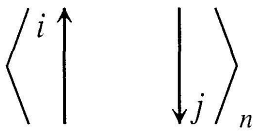
$=$  $+\left(-q^{1/2}[j]-q^{-1/2}[j]+[j+1]\right)$ 

$=$  $+\left([j-1]-q^{1/2}[j]-q^{-1/2}[j]+[j+1]\right)$ 

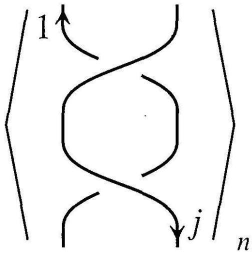
$=$  ,

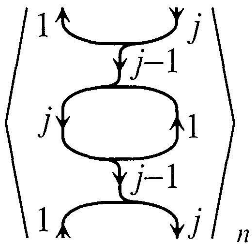
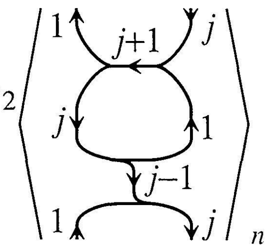
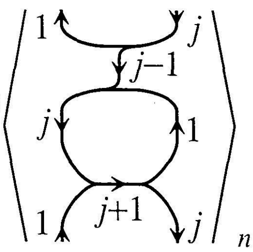
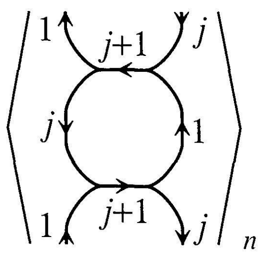
where we use Lemma A.4 in the third equality.

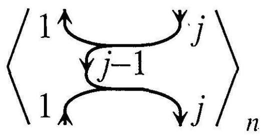
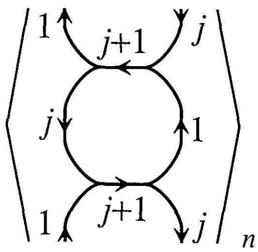
Next we will show

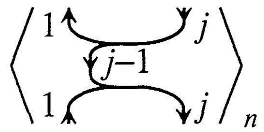
 $=$  .

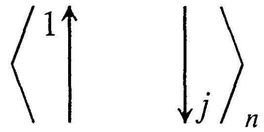
It also suffices to show the case $i=1$ as above. We have

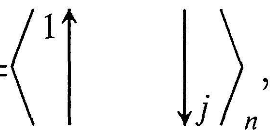


$=$  $-\,q^{1/2}$  $-\,q^{-1/2}$  $+$ 

$=\left([n-j+1]-q^{1/2}[n-j]-q^{-1/2}[n-j]\right)$  $+$ 

$$= \left([n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j] + [n-j-1]\right)$$ 

$+$ 

$=$  ,

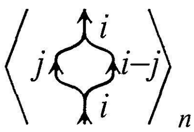
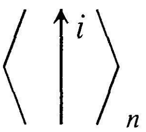
where we use Lemma 5.2 in the third equality. Now the proof for the Reidemeister move II is complete.

The invariance under the Reidemeister move III. This is proved by repeated application of Lemma 5.3 and details are omitted. See the proof of Theorem 3.1.

A. APPENDIX

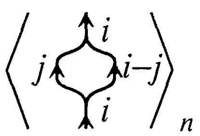
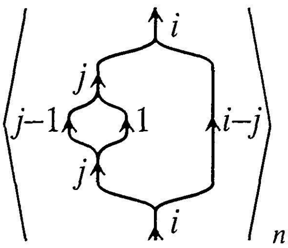
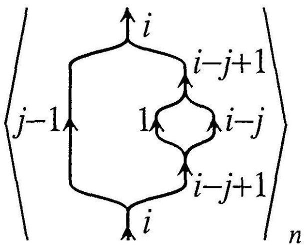
In this appendix, we give proofs of lemmas used in the previous section.

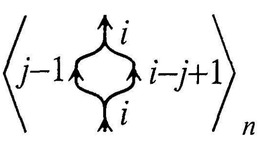
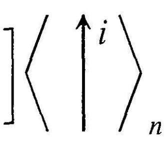
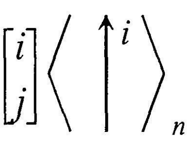
LEMMA A.1.

 $= \begin{bmatrix} i \\ j \end{bmatrix}$ 

for $i \geq j \geq 0$.

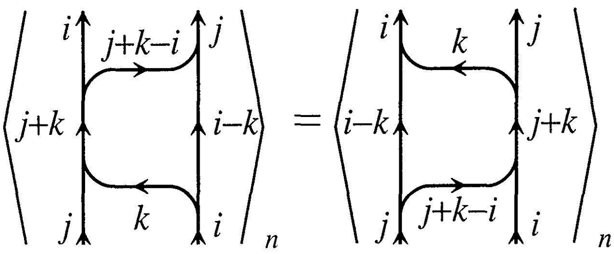
Proof. The proof for $j = 1$ is similar to that of Lemma 2.2 and omitted. For $j > 1$ we have

 $= \dfrac{1}{[j]}$  $= \dfrac{1}{[j]}$ 

$= \dfrac{[i-j+1]}{[j]}$  $= \dfrac{[i-j+1]}{[j]} \begin{bmatrix} i \\ j-1 \end{bmatrix}$  $= \begin{bmatrix} i \\ j \end{bmatrix}$  ,

where the second equality follows from Lemma 2.6 and the fourth by induction. The proof is complete. □

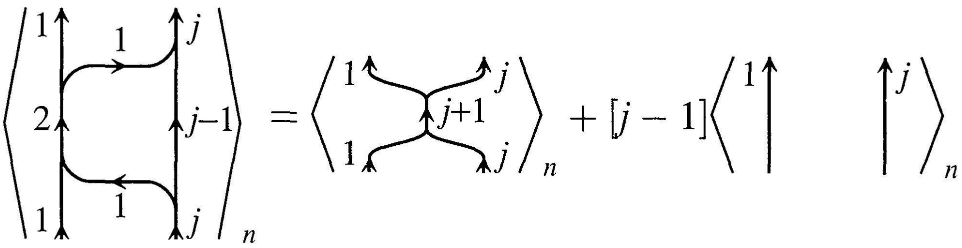
LEMMA A.2.



for $j+k \geq i \geq k \geq 0$.

Proof. By a π-rotation and orientation reversing, we get the right hand side from the left hand side. So there is a one to one correspondence between the states of both hand sides and the equality follows from the definition. □

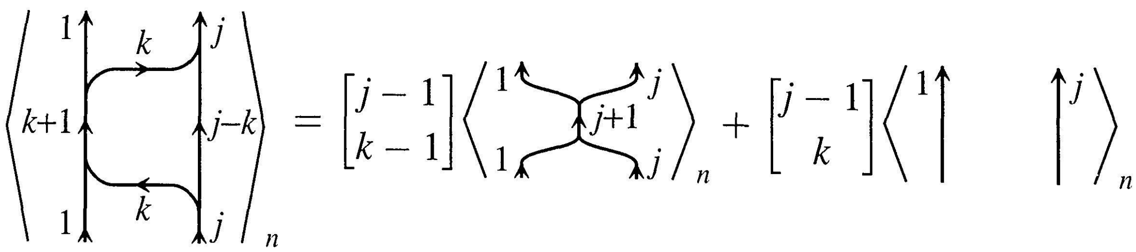
LEMMA A.3.



for $j \geq 1$.

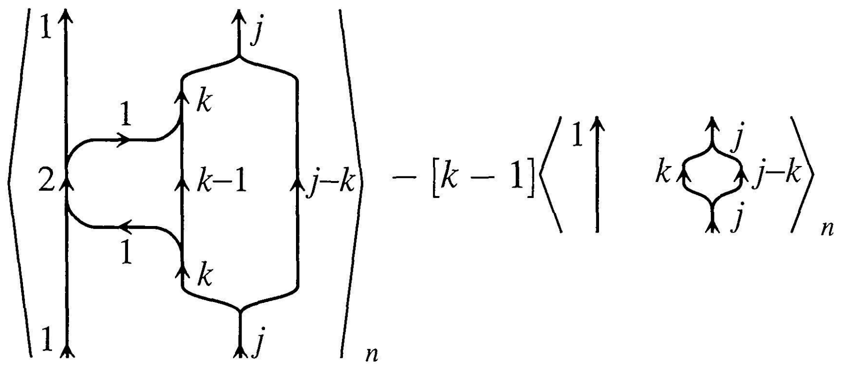
Proof. We can prove the equality directly from the definition as in the proofs in the lemmas in §2. Details are omitted. □

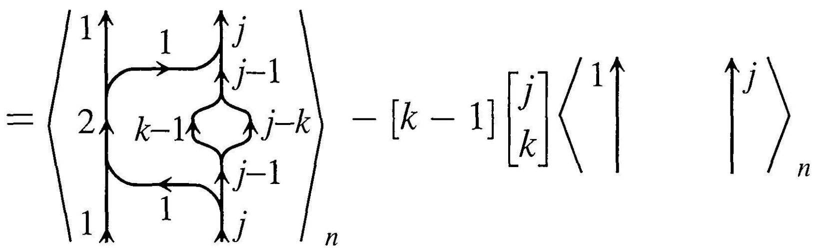
LEMMA A.4.

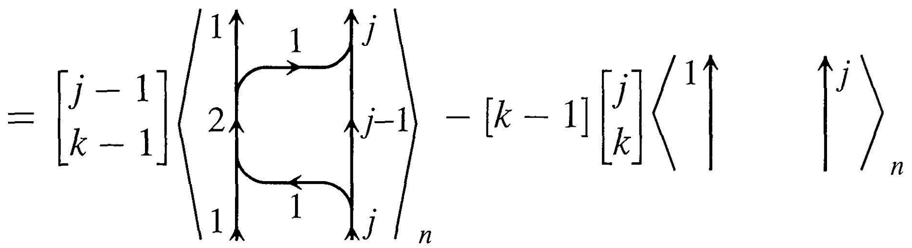


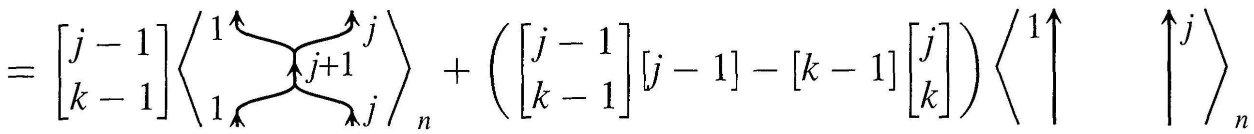
for $j \geq k \geq 1$.

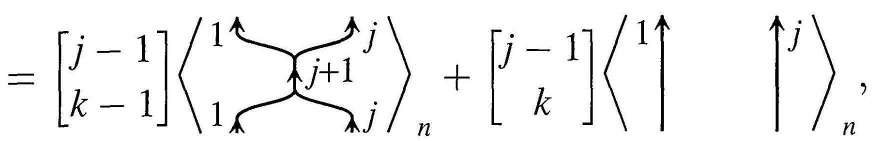
Proof. From Lemma A.3 (substituting j with k) the left hand side becomes





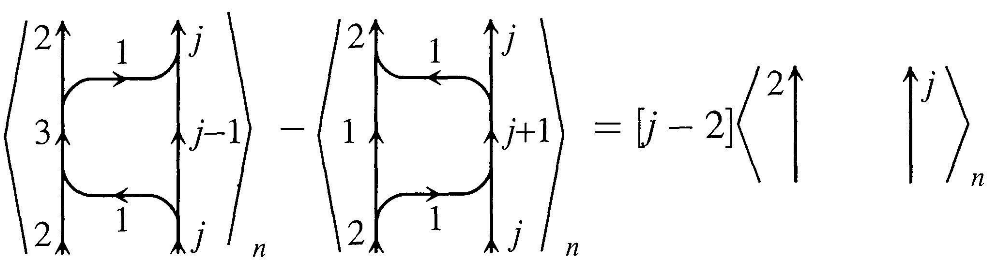






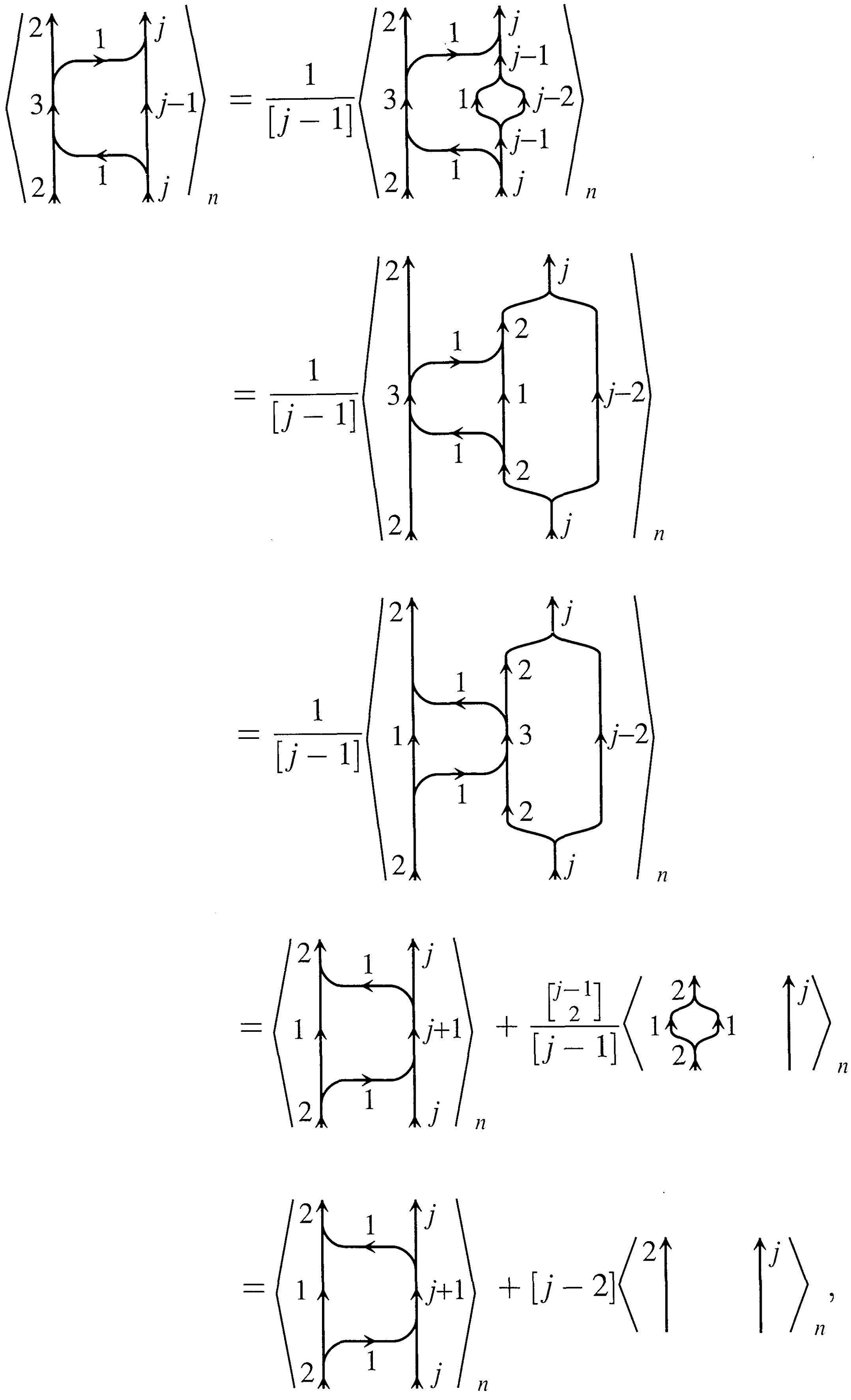
where the third equality follows from Lemma A.3. The proof is complete. $\square$

LEMMA A.5.



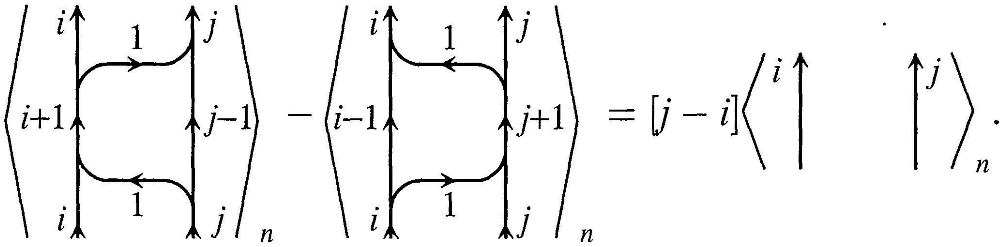
for $j \geq 2$.

Proof. We have

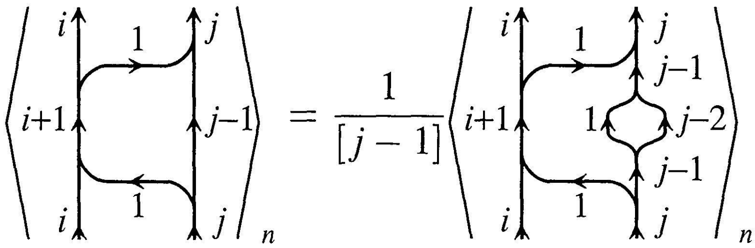


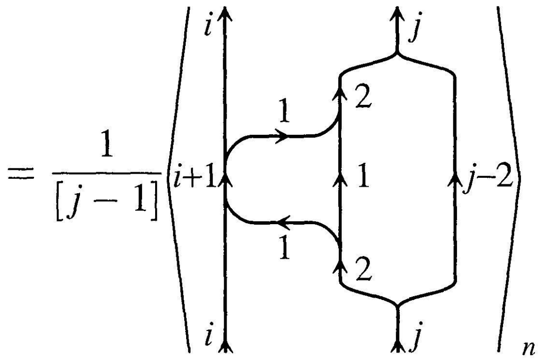
where we use Lemma A.2 in the third equality and Lemma A.4 in the fourth equality. The proof is complete. □

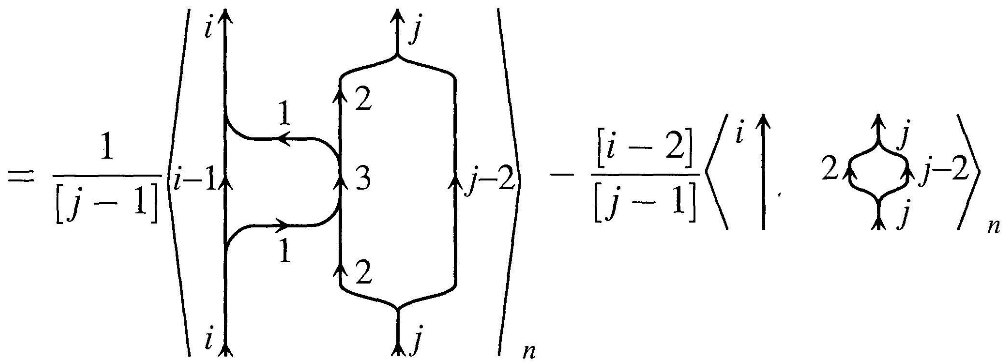
LEMMA A.6.

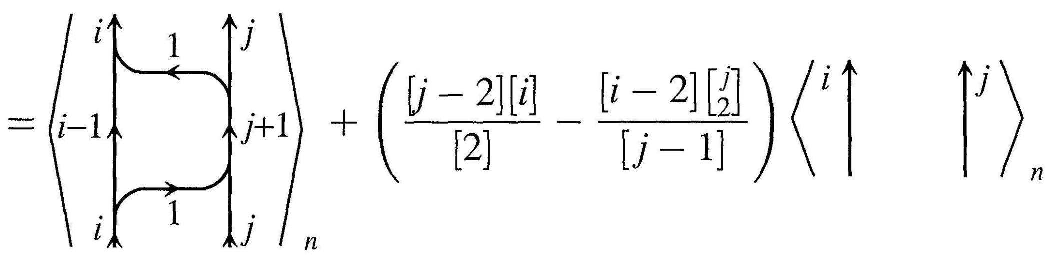


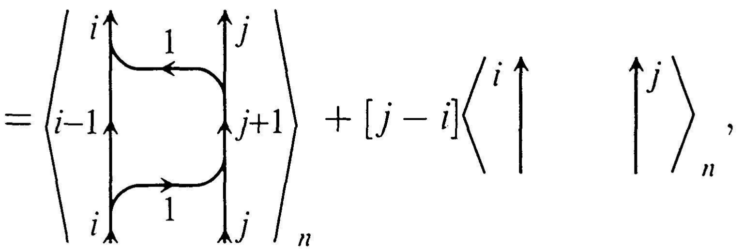
Proof. We have





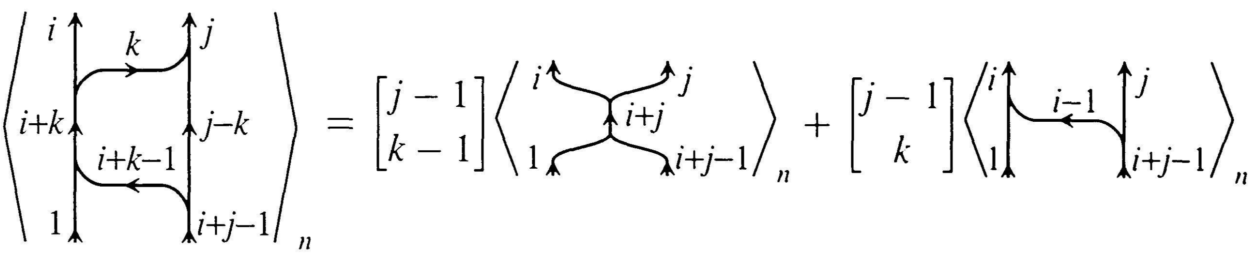






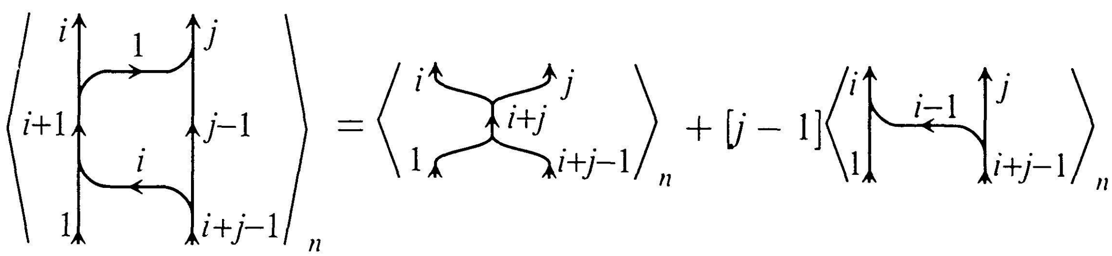
where we use Lemmas A.3 and A.4 in the third and the fourth equalities respectively. The proof is complete. □

LEMMA A.7.



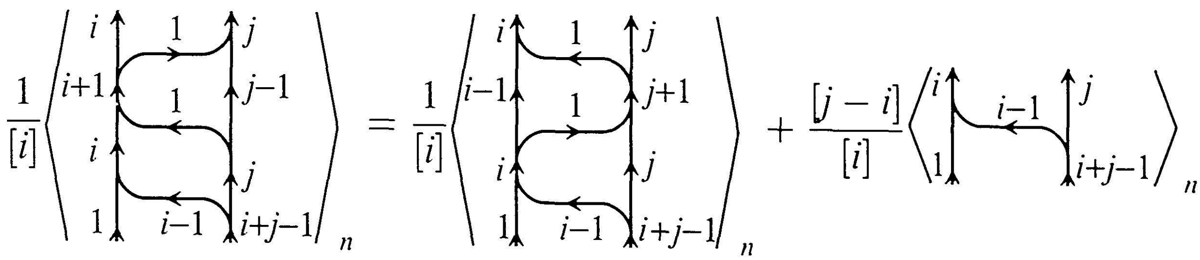
for $j > k \geq 1$.

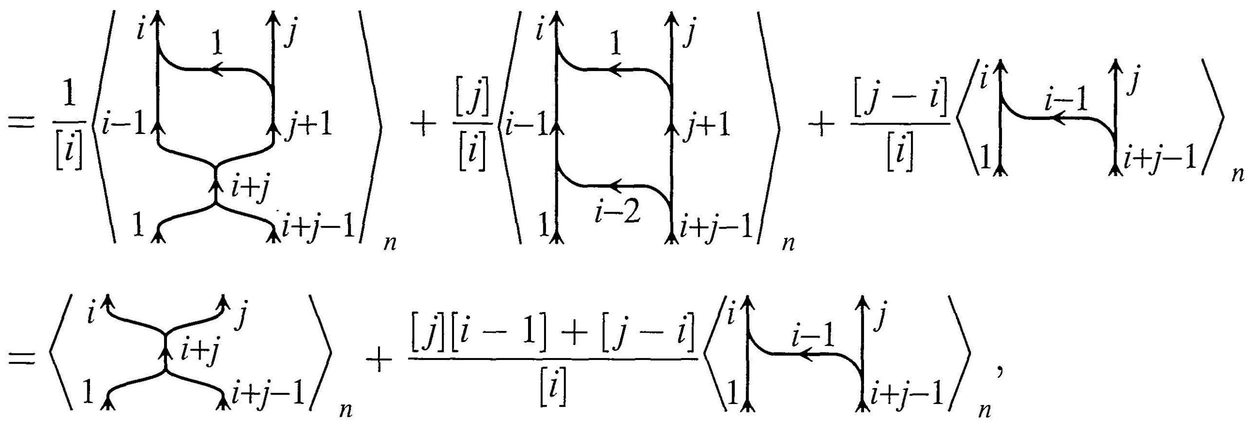
Proof. We first prove the case $k = 1$. We will show



by induction on j. Note that Lemma A.3 is the case $i = 1$.

The left hand side becomes

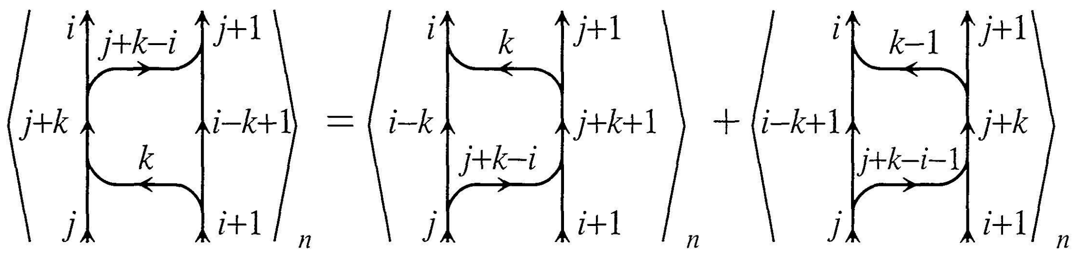




which is equal to the right hand side. Here we use Lemma A.6 in the first equality and the inductive hypothesis in the second equality.

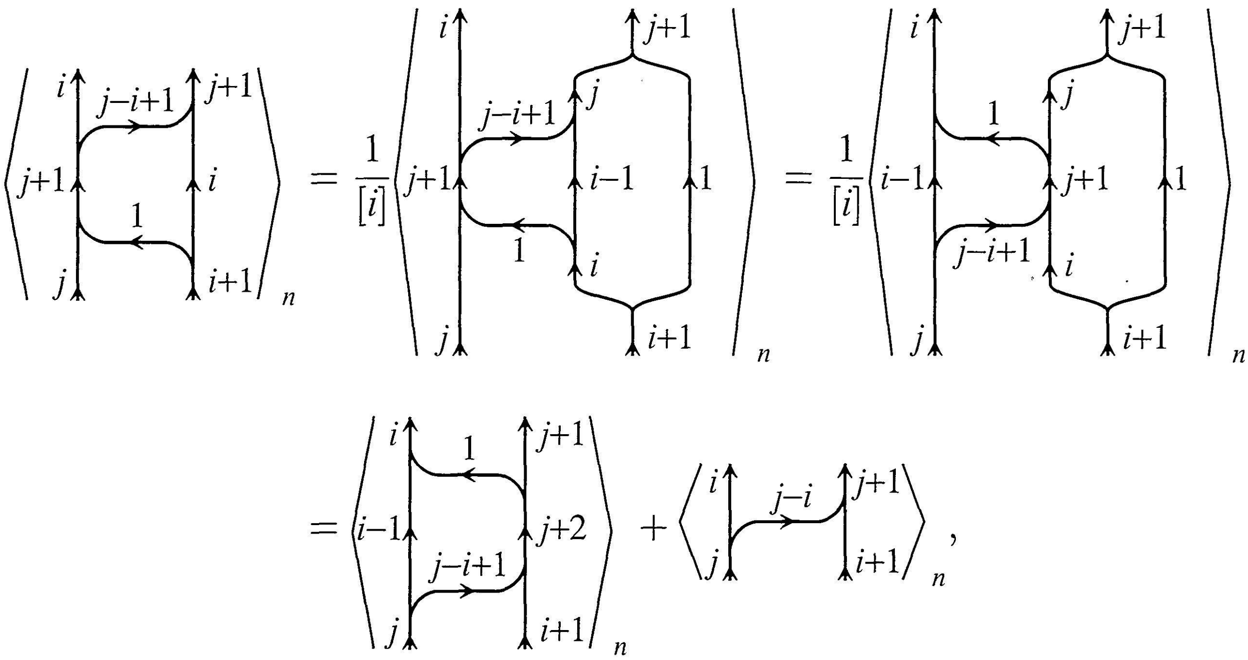
The proof for $k > 1$ is similar and omitted. □

LEMMA A.8.



for $j \geq i \geq k \geq 1$.

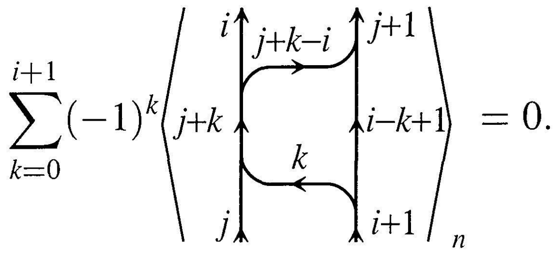
Proof. For $k = 1$ we have



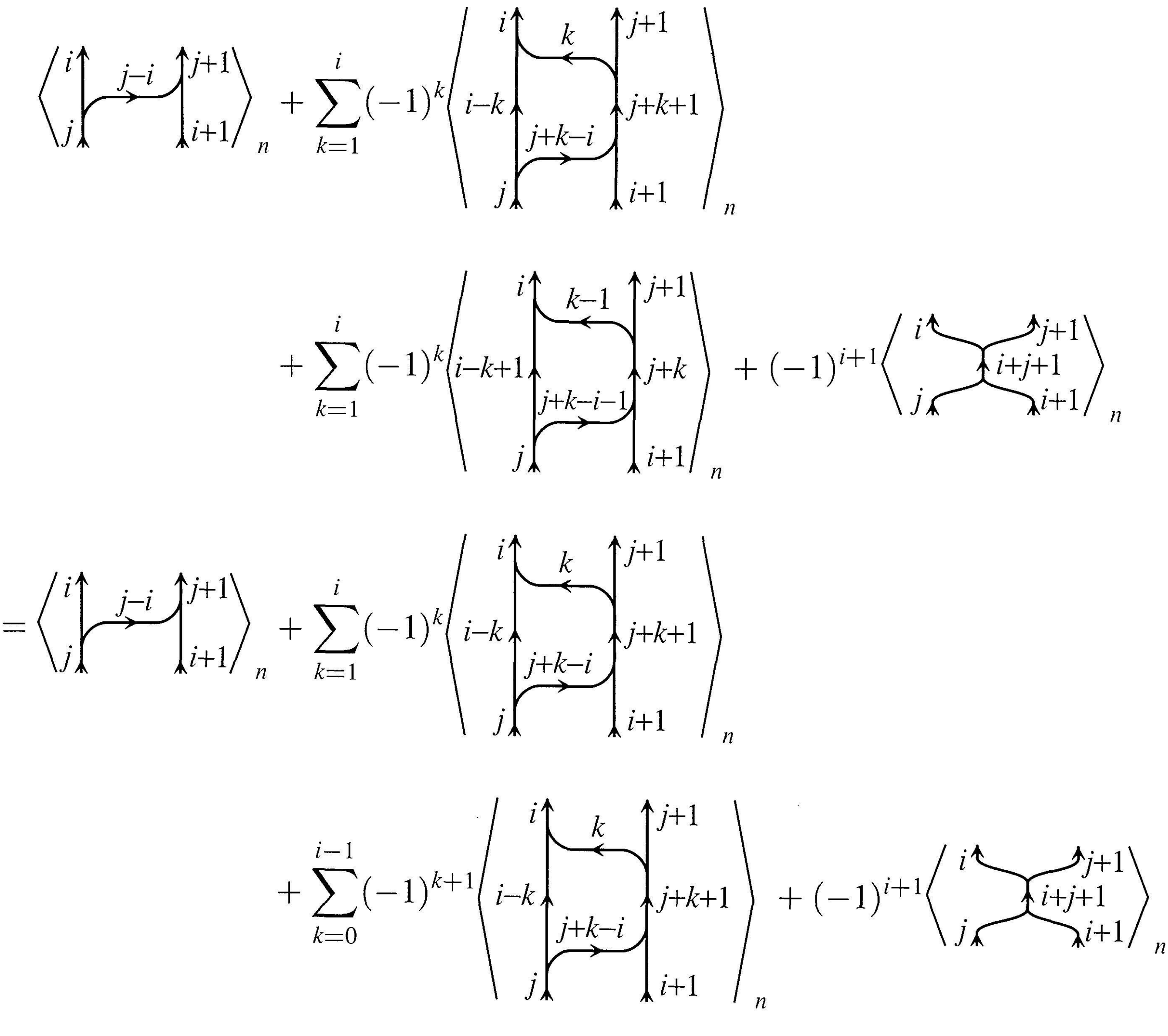
where we use Lemma A.7 in the last equality.

The proof for $k > 1$ is similar and left to the reader. □

Proposition A.9.



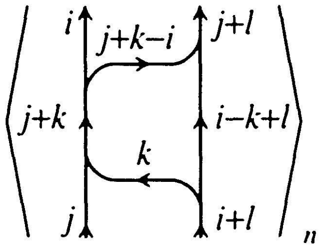
Proof. From Lemma A.8, we see that the left hand side equals



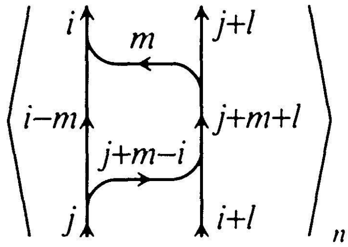
= 0, and the proof is complete. □

More generally we have the following formula, which was suggested by J. Murakami. The proof is left to the reader.

PROPOSITION A.10.



$$= \sum_{m=0}^{i} \begin{bmatrix} l \\ k-m \end{bmatrix}$$



for $j \geq i \geq k \geq 1$. *Here* $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ *if* $y < 0$ *or* $y > x$.

B. Table of the invariant

The following is the table of $P_n/[n]$ for knots with crossings less than or equal to eight. (Note that P_n is always divisible by $[n]$.) We refer the reader to [15] for the notations of knots. We used MAPLE V to tranlate the Homfly polynomial table in [10] to our invariants.

3_1	$-q^{2n}+q^{n+1}+q^{n-1}$

4_1	$q^n-q+1-q^{-1}+q^{-n}$

5_1	$-q^{3n+1}-q^{3n-1}+q^{2n+2}+q^{2n}+q^{2n-2}$
5_2	$-q^{3n}+q^{2n+1}-q^{2n}+q^{2n-1}+q^{n+1}-q^n+q^{n-1}$

6_1	$q^{2n}-q^{n+1}+q^n-q^{n-1}-q+2-q^{-1}+q^{-n}$
6_2	$q^{2n+1}-q^{2n}+q^{2n-1}-q^{n+2}+q^{n+1}-2q^n+q^{n-1}-q^{n-2}+q+q^{-1}$
6_3	$-q^{n+1}+q^n-q^{n-1}+q^2-q+3-q^{-1}+q^{-2}-q^{-n+1}+q^{-n}-q^{-n-1}$

7_1	$-q^{4n+2}-q^{4n}-q^{4n-2}+q^{3n+3}+q^{3n+1}+q^{3n-1}+q^{3n-3}$
7_2	$-q^{4n}+q^{3n+1}-q^{3n}+q^{3n-1}+q^{2n+1}-2q^{2n}+q^{2n-1}+q^{n+1}-q^n+q^{n-1}$
7_3	$q^{-2n+2}-q^{-2n+1}+q^{-2n}-q^{-2n-1}+q^{-2n-2}+q^{-3n+2}-q^{-3n+1}+2q^{-3n}$ $-q^{-3n-1}+q^{-3n-2}-q^{-4n+1}-q^{-4n-1}$
7_4	$q^{-n+1}-2q^{-n}+q^{-n-1}+2q^{-2n+1}-2q^{-2n}+2q^{-2n-1}+q^{-3n+1}-2q^{-3n}$ $+q^{-3n-1}-q^{-4n}$
7_5	$-q^{4n+1}+q^{4n}-q^{4n-1}+q^{3n+2}-2q^{3n+1}+2q^{3n}-2q^{3n-1}+q^{3n-2}+q^{2n+2}$ $-q^{2n+1}+2q^{2n}-q^{2n-1}+q^{2n-2}$
7_6	$-q^{3n}+2q^{2n+1}-2q^{2n}+2q^{2n-1}-q^{n+2}+2q^{n+1}-3q^n+2q^{n-1}-q^{n-2}$ $+q-1+q^{-1}$
7_7	$-q^{n+1}+2q^n-q^{n-1}+q^2-2q+4-2q^{-1}+q^{-2}-2q^{-n+1}+2q^{-n}$ $-2q^{-n-1}+q^{-2n}$

8_1	$q^{3n}-q^{2n+1}+q^{2n}-q^{2n-1}-q^{n+1}+2q^n-q^{n-1}-q+2-q^{-1}+q^{-n}$
8_2	$q^{3n+2}-q^{3n+1}+q^{3n}-q^{3n-1}+q^{3n-2}-q^{2n+3}+q^{2n+2}-2q^{2n+1}+q^{2n}$ $-2q^{2n-1}+q^{2n-2}-q^{2n-3}+q^{n+2}+q^n+q^{n-2}$
8_3	$q^{2n}-2q+3-2q^{-1}-q^{n+1}+2q^n-q^{n-1}-q^{-n+1}+2q^{-n}-q^{-n-1}+q^{-2n}$
8_4	$q^{2n+1}-q^{2n}+q^{2n-1}-q^{n+2}+2q^{n+1}-2q^n+2q^{n-1}-q^{n-2}-q^2+q-2$ $+q^{-1}-q^{-2}+q^{-n+1}+q^{-n-1}$
8_5	$q^{-n+2}+2q^{-n}+q^{-n-2}-q^{-2n+3}+q^{-2n+2}-3q^{-2n+1}+q^{-2n}-3q^{-2n-1}$ $+q^{-2n-2}-q^{-2n-3}+q^{-3n+2}-q^{-3n+1}+2q^{-3n}-q^{-3n-1}+q^{-3n-2}$
8_6	$q^{3n+1}-q^{3n}+q^{3n-1}-q^{2n+2}+2q^{2n+1}-3q^{2n}+2q^{2n-1}-q^{2n-2}-q^{n+2}$ $+2q^{n+1}-3q^n+2q^{n-1}-q^{n-2}+q+q^{-1}$
8_7	$-q^2+q-1+q^{-1}-q^{-2}+q^{-n+3}-q^{-n+2}+3q^{-n+1}-2q^{-n}+3q^{-n-1}$ $-q^{-n-2}+q^{-n-3}-q^{-2n+2}+q^{-2n+1}-2q^{-2n}+q^{-2n-1}-q^{-2n-2}$
8_8	$-q^{n+1}+q^n-q^{n-1}+q^2-2q+4-2q^{-1}+q^{-2}+q^{-n+2}-2q^{-n+1}$ $+3q^{-n}-2q^{-n-1}+q^{-n-2}-q^{-2n+1}+q^{-2n}-q^{-2n-1}$
8_9	$q^{n+2}-q^{n+1}+2q^n-q^{n-1}+q^{n-2}-q^3+q^2-3q+3-3q^{-1}+q^{-2}-q^{-3}$ $+q^{-n+2}-q^{-n+1}+2q^{-n}-q^{-n-1}+q^{-n-2}$
8_{10}	$-q^2+q-2+q^{-1}-q^{-2}+q^{-n+3}-q^{-n+2}+4q^{-n+1}-2q^{-n}+4q^{-n-1}$ $-q^{-n-2}+q^{-n-3}-q^{-2n+2}+q^{-2n+1}-3q^{-2n}+q^{-2n-1}-q^{-2n-2}$
8_{11}	$q^{3n+1}-q^{3n}+q^{3n-1}-q^{2n+2}+2q^{2n+1}-4q^{2n}+2q^{2n-1}-q^{2n-2}+q-1$ $+q^{-1}-q^{n+2}+3q^{n+1}-3q^n+3q^{n-1}-q^{n-2}$
8_{12}	$-2q^{n+1}+q^{2n}-2q^{n-1}+3q^n+q^2-3q+5-3q^{-1}+q^{-2}-2q^{-n+1}$ $+3q^{-n}-2q^{-n-1}+q^{-2n}$
8_{13}	$-q^{2n+1}+q^{2n}-q^{2n-1}+q^{n+2}-2q^{n+1}+4q^n-2q^{n-1}+q^{n-2}-q^{-n+1}$ $+2q^{-n}-q^{-n-1}+q^2-3q+4-3q^{-1}+q^{-2}$
8_{14}	$q^{3n+1}-2q^{3n}+q^{3n-1}-q^{2n+2}+3q^{2n+1}-4q^{2n}+3q^{2n-1}-q^{2n-2}-q^{n+2}$ $+3q^{n+1}-4q^n+3q^{n-1}-q^{n-2}+q-1+q^{-1}$
8_{15}	$q^{5n}-3q^{4n+1}+2q^{4n}-3q^{4n-1}+2q^{3n+2}-3q^{3n+1}+5q^{3n}-3q^{3n-1}+2q^{3n-2}$ $+q^{2n+2}-2q^{2n+1}+3q^{2n}-2q^{2n-1}+q^{2n-2}$
8_{16}	$-q^{2n+2}+2q^{2n+1}-3q^{2n}+2q^{2n-1}-q^{2n-2}+q^{n+3}-2q^{n+2}+4q^{n+1}-4q^n$ $+4q^{n-1}-2q^{n-2}+q^{n-3}-q^2+2q-2+2q^{-1}-q^{-2}$

8_{17}	$q^{n+2} - 2q^{n+1} + 3q^{n} - 2q^{n-1} + q^{n-2} - q^{3} + 2q^{2} - 4q + 5 - 4q^{-1} + 2q^{-2} - q^{-3} + q^{-n+2} - 2q^{-n+1} + 3q^{-n} - 2q^{-n-1} + q^{-n-2}$
8_{18}	$q^{n+2} - 3q^{n+1} + 3q^{n} - 3q^{n-1} + q^{n-2} - q^{3} + 3q^{2} - 4q + 7 - 4q^{-1} + 3q^{-2} - q^{-3} + q^{-n+2} - 3q^{-n+1} + 3q^{-n} - 3q^{-n-1} + q^{-n-2}$
8_{19}	$q^{-3n+3} + q^{-3n+1} + q^{-3n} + q^{-3n-1} + q^{-3n-3} - q^{-4n+2} - q^{-4n+1} - q^{-4n} - q^{-4n-1} - q^{-4n-2} + q^{-5n}$
8_{20}	$-q^{2n+1} - q^{2n-1} + q^{n+2} + 2q^{n} + q^{n-2} - q + 1 - q^{-1}$
8_{21}	$q^{3n+1} - q^{3n} + q^{3n-1} - q^{2n+2} + q^{2n+1} - 3q^{2n} + q^{2n-1} - q^{2n-2} + 2q^{n+1} - q^{n} + 2q^{n-1}$

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