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HOMFLY POLYNOMIAL VIA AN INVARIANT OF COLORED PLANE GRAPHS

by Hitoshi MURAKAMI, Tomotada OHTSUKI and Shuji YAMADA

ABSTRACT. After the first discovery of the quantum invariant associated with $SU(2)$ by V.F.R. Jones [3], the invariants associated with $SU(n)$ were found by several authors [1]. It was first proved by V.G. Turaev [16] that all these come from so-called "quantum groups", especially from their R-matrices corresponding to the vector representations. There also exist various quantum invariants corresponding to other representations (see for example [7], [14], [11]).

The aim of this paper is to give a graphical way to define $SU(n)$ quantum invariants for links. To do this we first construct an invariant of colored, oriented, trivalent, plane graphs for each n (≥ 2). Then we show that the $SU(n)$ polynomial invariant corresponding to the vector representation (HOMFLY polynomial) can be defined by using our graph invariant.

We can also show that our invariant defines the $SU(n)$ polynomial invariant corresponding to the anti-symmetric tensors of the vector representation.

We note that our graph invariant for $SU(3)$ was first introduced by G. Kuperberg in [8]. The second and the third authors used it in [12] to construct magic elements and defined the quantum $SU(3)$ invariants for 3-manifolds. Now [12] and the present paper together give an elementary and self-contained proof of the existence of magic elements for $SU(3)$ and so that of the quantum $SU(3)$ invariants of 3-manifolds just like W.B.R. Lickorish did for $SU(2)$ in [9] using the Kauffman bracket [5]. See [17] for a similar approach to $SU(n)$ invariants of 3-manifolds.

We also note that our graph invariant may be obtained (not checked yet) by direct computations of the universal R-matrix. But the advantage of our definition is that it does not require any knowledge of quantum groups nor representation theory. On the contrary we can recover the R-matrix of the quantum group $U_q(\mathfrak{sl}(n,\mathbf{C}))$ corresponding at least to the vector representation.

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