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Artikel: HOMFLY POLYNOMIAL VIA AN INVARIANT OF COLORED PLANE GRAPHS

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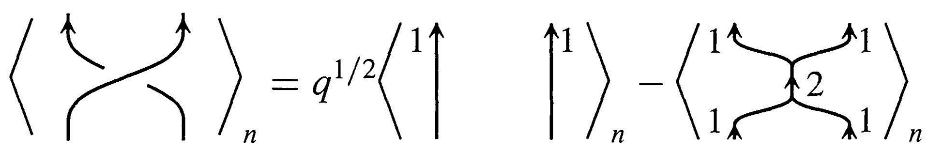
§ 3. Invariants of links

In this section, we give a graphical definition of the HOMFLY polynomial of oriented links. For an oriented link diagram D, we define $\langle D \rangle_n$ by the following.



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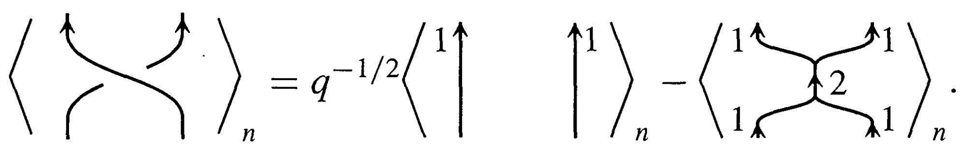
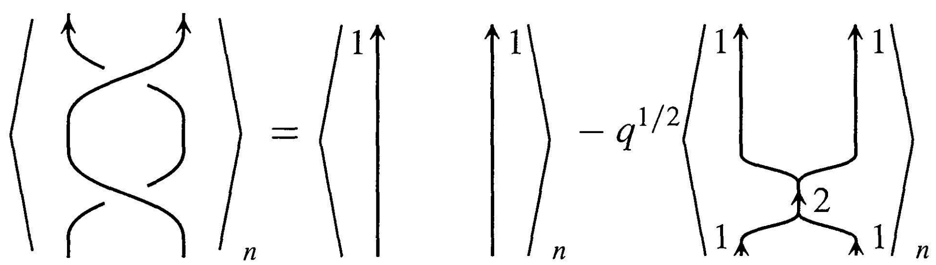
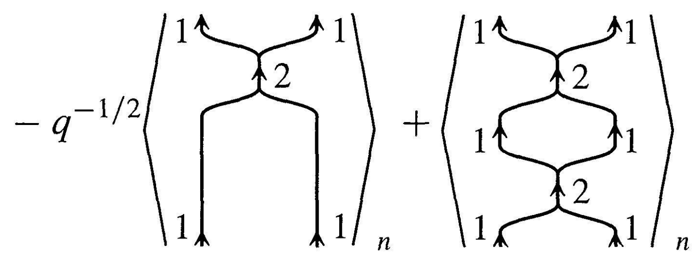


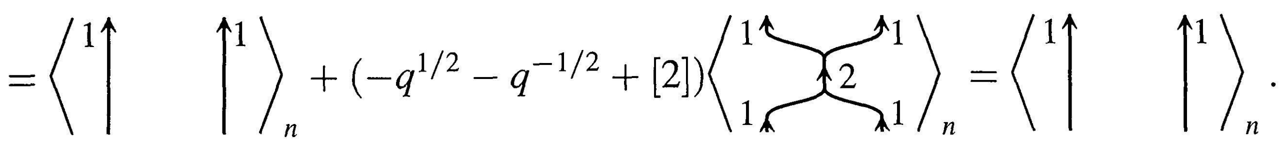
Then we have

THEOREM 3.1. *$\langle D \rangle_n$ is invariant under the Reidemeister moves II and III.*

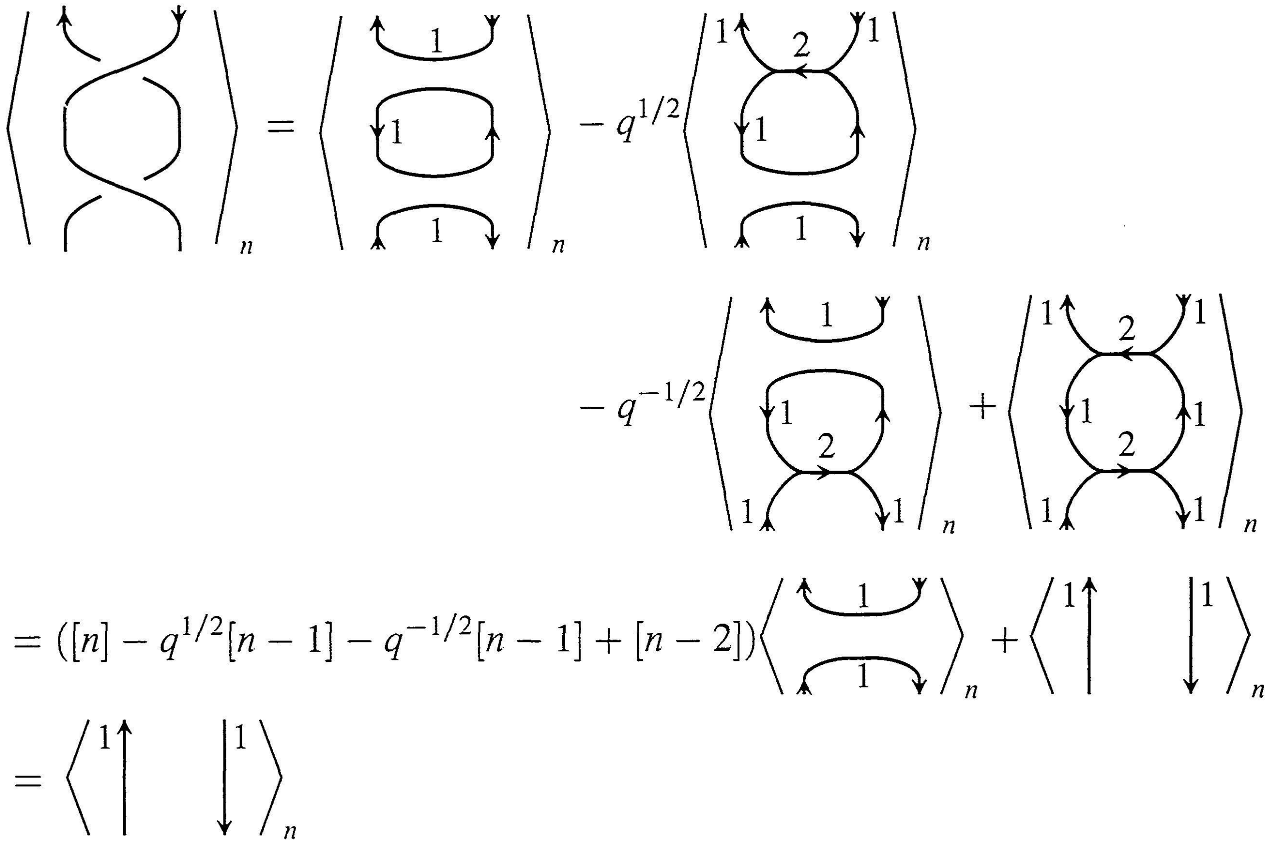
Proof. From Lemma 2.2, we have



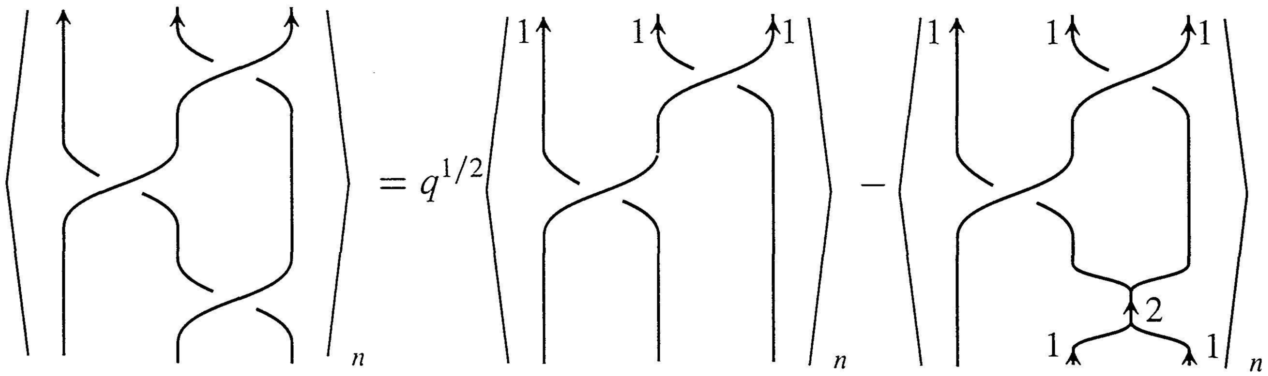




We also have



from Lemmas 2.1, 2.3 and 2.4 and so $\langle D\rangle_n$ is invariant under the Reidemeister move II. Next we prove the invariance under the Reidemeister move III. Since we have



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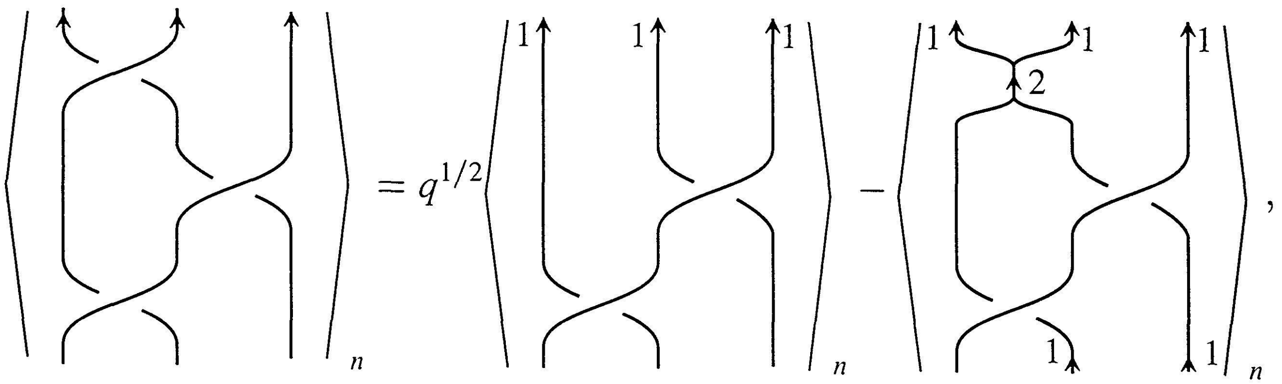
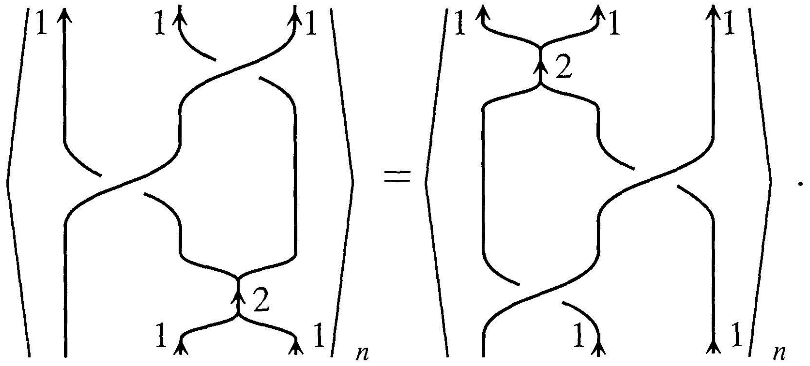
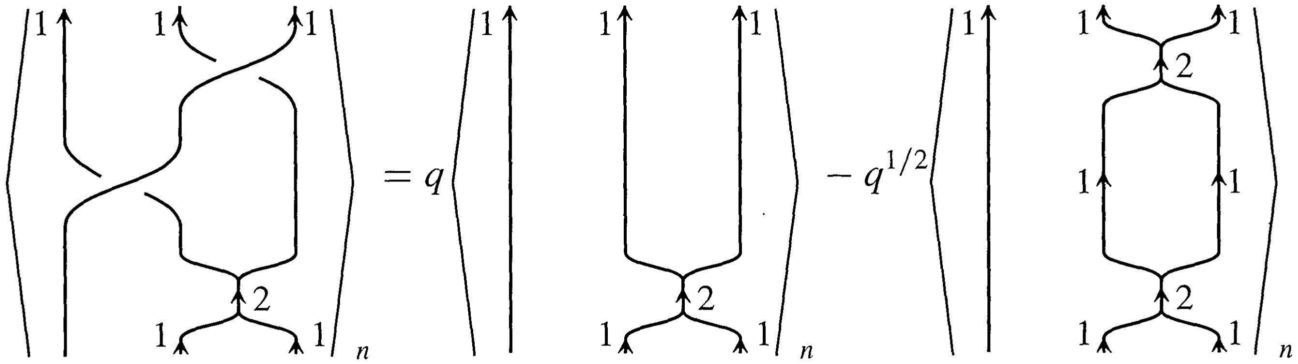
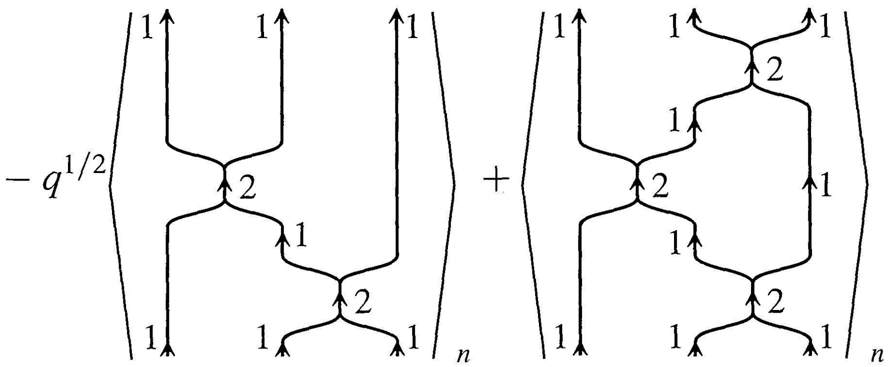


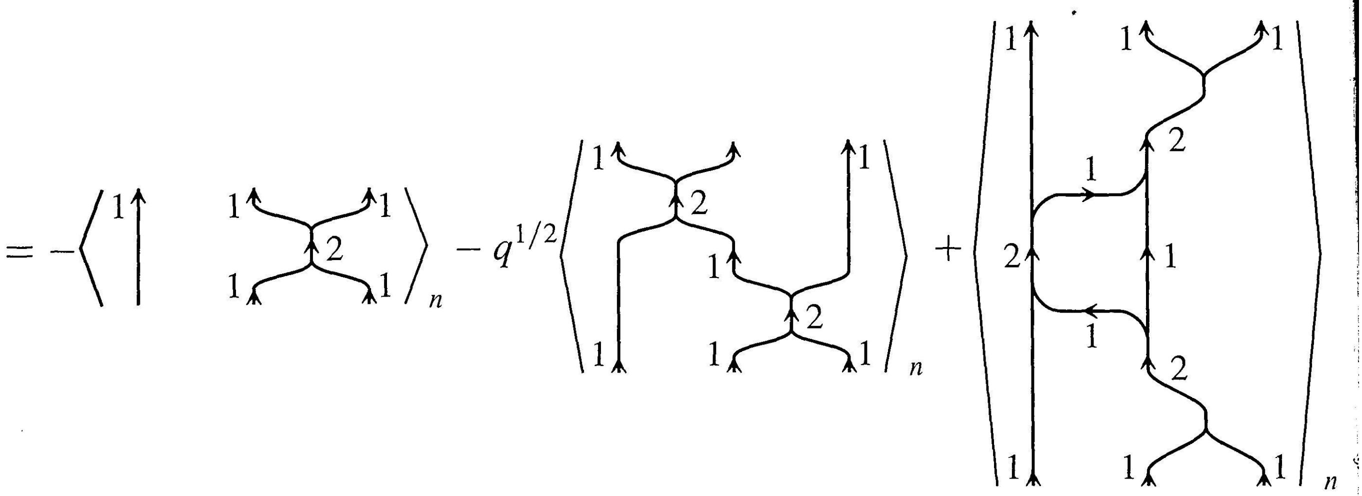
it suffices to show that

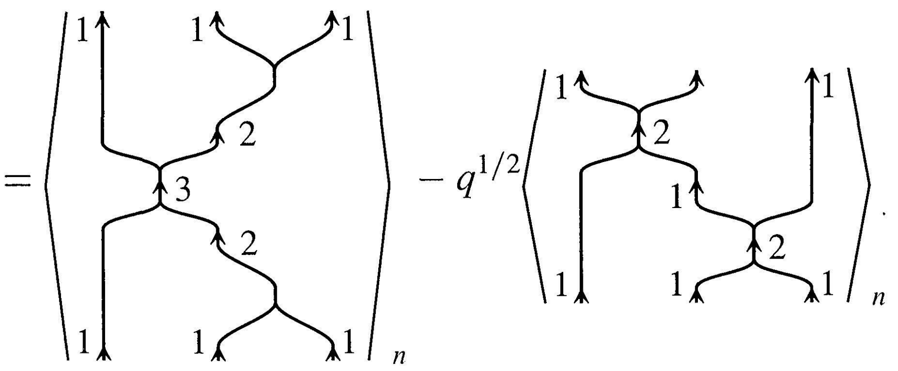


Since

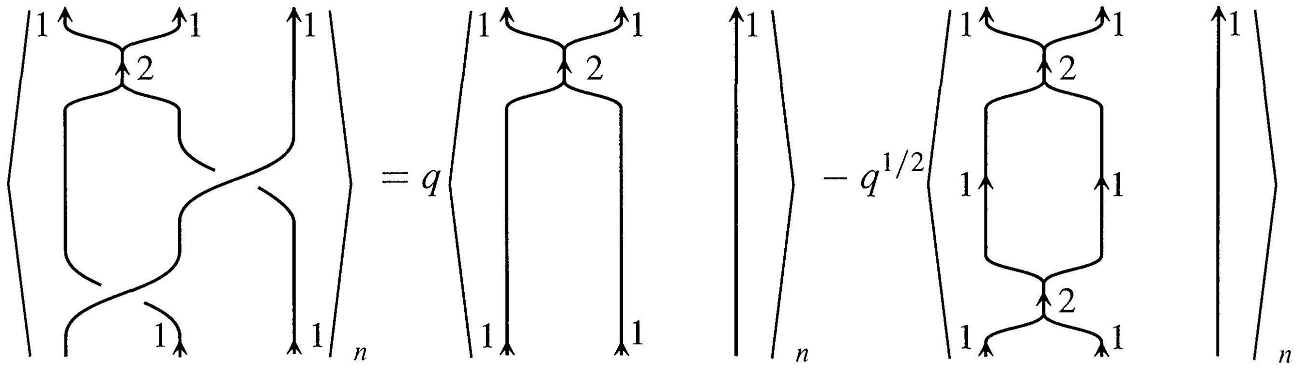


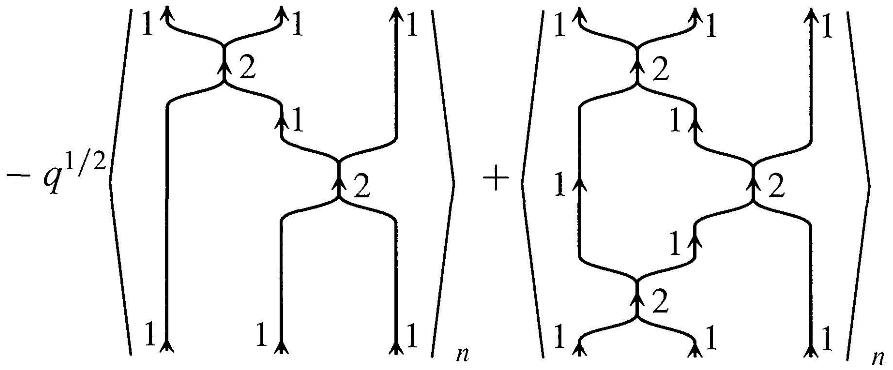


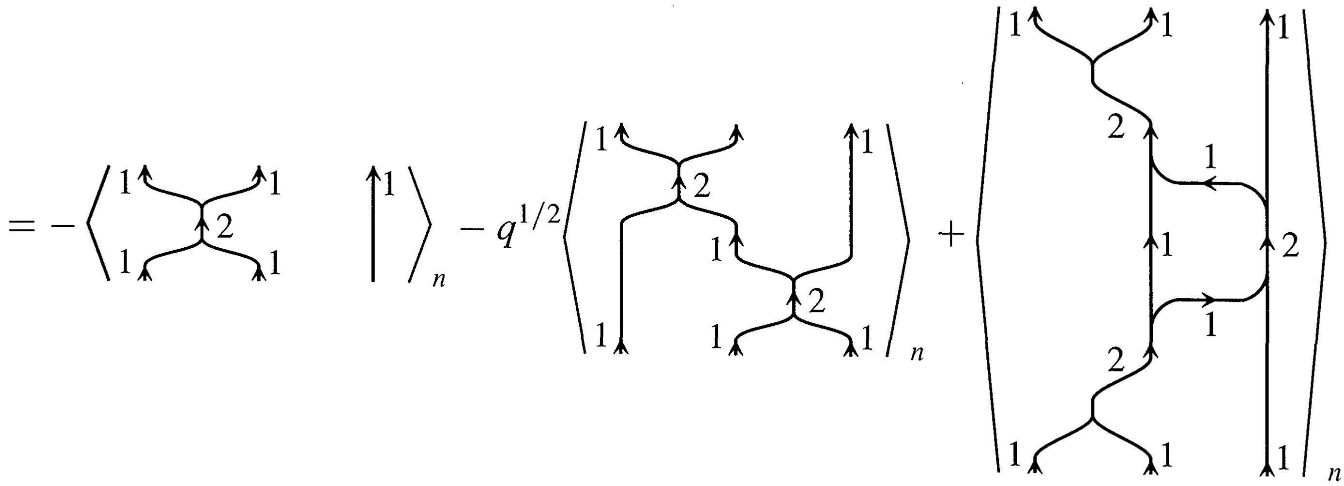


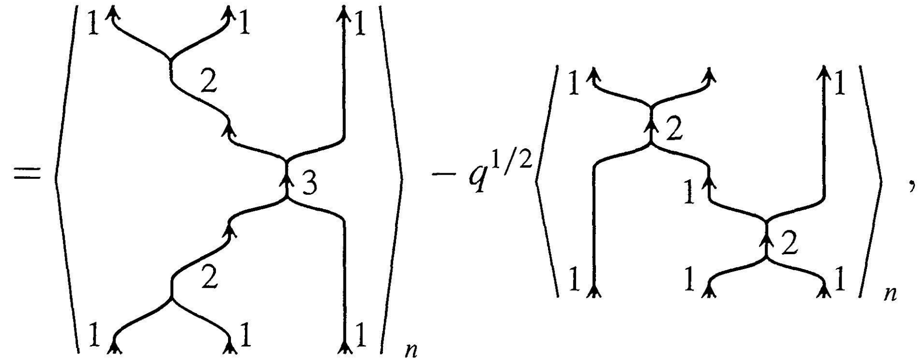


from Lemmas 2.2 and 2.5 and









we have the required formula from Lemma 2.6.

Since the Reidemeister move III with other types of orientations can be obtained from the Reidemeister moves II and III described above (see [12]), the proof is complete. □

If we define $P_n(D) = q^{(n/2)(-\mathrm{w}(D))}\langle D\rangle_n$ with w(D) the writhe (the algebraic sum of the crossings) of D, then we have

THEOREM 3.2. *$P_n(D)$ is invariant under the Reidemeister moves I, II, and III and satisfies the following skein relation.*

$$q^{n/2}P_n(D_+) - q^{-n/2}P_n(D_-) = (q^{1/2} - q^{-1/2})P_n(D_0),$$

where D_+, D_- and D_0 are identical link diagrams except near a crossing as described in Figure 3.1.

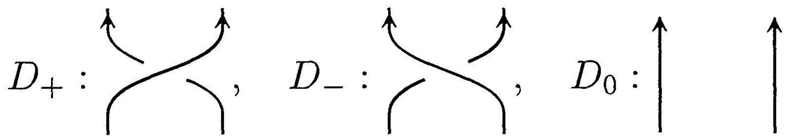
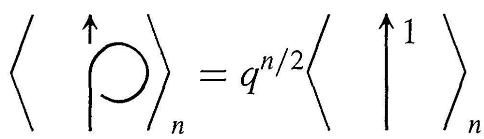


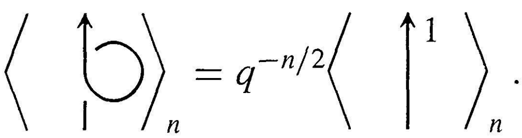
FIGURE 3.1
skein triple

The proof of this theorem follows immediately from the following lemma.

LEMMA 3.3.



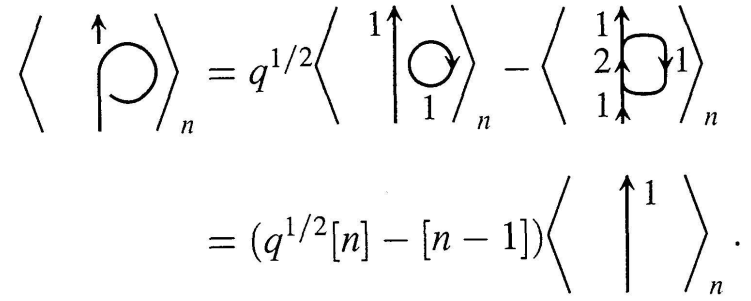
and



Proof. From Lemmas 2.1 and 2.3, we have

(3.1)

(3.2)



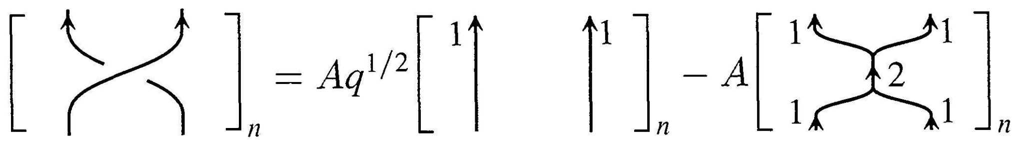
Since

$$q^{1/2}[n] - [n-1] = q^{n/2},$$

the first equality follows. The second equality follows similarly and the proof is complete. □

Therefore P_n defines a link invariant and so we can put $P_n(L) = P_n(D)$ for the link L presented by D. Then P_n is a version of the HOMFLY polynomial [1], [13].

REMARK 3.4. If we define $[D]_n$ to be



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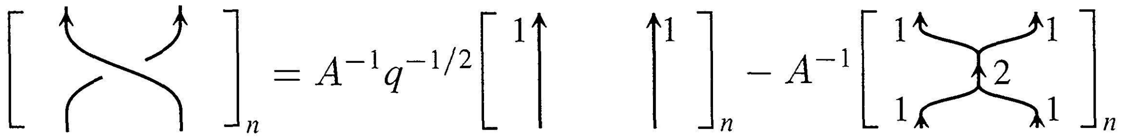

with A an indeterminate, we also have a framed link invariant.

When $n = 2$ and $A = q^{-1/4}$, we have a version of the Kauffman bracket naturally defined from representation theory of $U_q(\mathfrak{sl}(2, \mathbf{C}))$ [6, Theorem 4.3].

For $n = 3$ we have G. Kuperberg's recursive formula [8] as follows. Consider an oriented, trivalent, plane graph with flow less than or equal to three. If we reverse the orientations of all the edges with flow two, remove all the edges with flow three, and forget the flow, then we have a trivalent graph without flow such that at every vertex all the edges are in or out. Putting $A = q^{-1/6}$ and replacing q with q^{-1}, we have Kuperberg's formula. (This corresponds to the fact that the two-fold anti-symmetric tensor of the vector representation of $SU(3)$ is isomorphic to its dual.) See also [12].