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$$\begin{aligned}
 &= \left([n-j+1] - q^{1/2}[n-j] - q^{-1/2}[n-j] + [n-j-1] \right) \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \\ 1 \uparrow \end{array} \right\rangle_n \\
 &\quad + \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} 1 \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \end{array} \right\rangle_n,
 \end{aligned}$$

where we use Lemma 5.2 in the third equality. Now the proof for the Reidemeister move II is complete.

The invariance under the Reidemeister move III. This is proved by repeated application of Lemma 5.3 and details are omitted. See the proof of Theorem 3.1.

A. APPENDIX

In this appendix, we give proofs of lemmas used in the previous section.

LEMMA A.1.

$$\left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \quad \downarrow i-j \\ \uparrow i \end{array} \right\rangle_n = [j] \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \end{array} \right\rangle_n$$

for $i \geq j \geq 0$.

Proof. The proof for $j = 1$ is similar to that of Lemma 2.2 and omitted. For $j > 1$ we have

$$\begin{aligned}
 \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \quad \downarrow i-j \\ \uparrow i \end{array} \right\rangle_n &= \frac{1}{[j]} \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j-1 \quad \downarrow 1 \\ \uparrow j \quad \uparrow i-j \end{array} \right\rangle_n = \frac{1}{[j]} \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j-1 \quad \downarrow 1 \\ \uparrow i \quad \uparrow i-j+1 \end{array} \right\rangle_n \\
 &= \frac{[i-j+1]}{[j]} \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j-1 \quad \downarrow i-j+1 \\ \uparrow i \end{array} \right\rangle_n = \frac{[i-j+1]}{[j]} [j-1] \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j-1 \end{array} \right\rangle_n = [j] \left\langle \begin{array}{c} \uparrow i \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \downarrow j \end{array} \right\rangle_n,
 \end{aligned}$$

where the second equality follows from Lemma 2.6 and the fourth by induction. The proof is complete. \square

LEMMA A.2.

$$\left\langle \begin{array}{c} i \\ j+k-i \\ j \\ i-k \\ j+k \\ j \\ k \\ i \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \\ j \\ i-k \\ j+k \\ j \\ j+k-i \\ i \end{array} \right\rangle_n$$

for $j+k \geq i \geq k \geq 0$.

Proof. By a π -rotation and orientation reversing, we get the right hand side from the left hand side. So there is a one to one correspondence between the states of both hand sides and the equality follows from the definition. \square

LEMMA A.3.

$$\left\langle \begin{array}{c} 1 \\ j \\ 2 \\ j-1 \\ 1 \\ j \end{array} \right\rangle_n = \left\langle \begin{array}{c} 1 \\ j \\ 1 \\ j+1 \\ j \end{array} \right\rangle_n + [j-1] \left\langle \begin{array}{c} 1 \\ j \end{array} \right\rangle_n$$

for $j \geq 1$.

Proof. We can prove the equality directly from the definition as in the proofs in the lemmas in §2. Details are omitted. \square

LEMMA A.4.

$$\left\langle \begin{array}{c} 1 \\ j \\ k+1 \\ j-k \\ 1 \\ j \\ k \end{array} \right\rangle_n = \left[\begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \\ j \\ 1 \\ j+1 \\ j \end{array} \right\rangle_n + \left[\begin{array}{c} j-1 \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \\ j \end{array} \right\rangle_n$$

for $j \geq k \geq 1$.

Proof. From Lemma A.3 (substituting j with k) the left hand side becomes

$$\begin{aligned}
 & \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow \begin{array}{c} \begin{array}{c} \uparrow k \\ \uparrow k-1 \\ \uparrow j-k \end{array} \\ \begin{array}{c} \uparrow k \\ \uparrow j \end{array} \end{array} \\ 1 \uparrow \end{array} \right\rangle_n - [k-1] \left\langle \begin{array}{c} 1 \uparrow \\ \begin{array}{c} \uparrow j \\ \uparrow j \end{array} \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow \begin{array}{c} \begin{array}{c} \uparrow j \\ \uparrow j-1 \\ \uparrow j-k \end{array} \\ \begin{array}{c} \uparrow j-1 \\ \uparrow j \end{array} \end{array} \\ 1 \uparrow \end{array} \right\rangle_n - [k-1] \left[\begin{array}{c} j \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[\begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ 2 \leftarrow \begin{array}{c} \uparrow j \\ \uparrow j-1 \end{array} \\ 1 \uparrow \end{array} \right\rangle_n - [k-1] \left[\begin{array}{c} j \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[\begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \begin{array}{c} \uparrow j+1 \\ \uparrow j \end{array} \end{array} \right\rangle_n + \left(\left[\begin{array}{c} j-1 \\ k-1 \end{array} \right] [j-1] - [k-1] \left[\begin{array}{c} j \\ k \end{array} \right] \right) \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n \\
 &= \left[\begin{array}{c} j-1 \\ k-1 \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \begin{array}{c} \uparrow j+1 \\ \uparrow j \end{array} \end{array} \right\rangle_n + \left[\begin{array}{c} j-1 \\ k \end{array} \right] \left\langle \begin{array}{c} 1 \uparrow \\ \uparrow j \end{array} \right\rangle_n,
 \end{aligned}$$

where the third equality follows from Lemma A.3. The proof is complete. \square

LEMMA A.5.

$$\left\langle \begin{array}{c} 2 \uparrow \\ 3 \leftarrow \begin{array}{c} \uparrow j \\ \uparrow j-1 \end{array} \\ 2 \uparrow \end{array} \right\rangle_n - \left\langle \begin{array}{c} 2 \uparrow \\ 1 \leftarrow \begin{array}{c} \uparrow j \\ \uparrow j+1 \end{array} \\ 2 \uparrow \end{array} \right\rangle_n = [j-2] \left\langle \begin{array}{c} 2 \uparrow \\ \uparrow j \end{array} \right\rangle_n$$

for $j \geq 2$.

Proof. We have

$$\begin{aligned}
 & \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{j-1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n = \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j-1 \uparrow \\ j-2 \uparrow \\ j-1 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 & = \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 & = \frac{1}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ 2 \uparrow \\ 3 \uparrow \\ 2 \uparrow \\ j \uparrow \end{array} \right\rangle_n \\
 & = \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + \frac{[j-1]}{[j-1]} \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \end{array} \right\rangle_n \\
 & = \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + [j-2] \left\langle \begin{array}{c} 2 \uparrow \\ 1 \uparrow \\ 2 \uparrow \end{array} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1} \\ \xrightarrow{1} \end{array} \begin{array}{c} j \uparrow \end{array} \right\rangle_n,
 \end{aligned}$$

where we use Lemma A.2 in the third equality and Lemma A.4 in the fourth equality. The proof is complete. \square

LEMMA A.6.

$$\left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ \downarrow i \\ j \uparrow \\ j-1 \uparrow \end{array} \right\rangle_n - \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ \downarrow i \\ j \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n = [j-i] \left\langle \begin{array}{c} i \uparrow \\ \uparrow j \end{array} \right\rangle_n.$$

Proof. We have

$$\begin{aligned} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ \downarrow i \\ j \uparrow \\ j-1 \uparrow \end{array} \right\rangle_n &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ \downarrow i \\ j \uparrow \\ j-1 \uparrow \\ \downarrow j-2 \end{array} \right\rangle_n \\ &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ \downarrow i \\ j \uparrow \\ 2 \uparrow \\ 1 \uparrow \\ \downarrow 2 \\ j-2 \uparrow \end{array} \right\rangle_n \\ &= \frac{1}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ \downarrow i \\ j \uparrow \\ 2 \uparrow \\ 3 \uparrow \\ \downarrow 2 \\ j-2 \uparrow \end{array} \right\rangle_n - \frac{[i-2]}{[j-1]} \left\langle \begin{array}{c} i \uparrow \\ \uparrow j \\ \downarrow j \\ \uparrow j-2 \end{array} \right\rangle_n \\ &= \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ \downarrow i \\ j \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n + \left(\frac{[j-2][i]}{[2]} - \frac{[i-2][j]}{[j-1]} \right) \left\langle \begin{array}{c} i \uparrow \\ \uparrow j \end{array} \right\rangle_n \end{aligned}$$

$$= \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \\ j \uparrow \\ j+1 \uparrow \\ j \uparrow \end{array} \right\rangle_n + [j-i] \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \end{array} \right\rangle_n,$$

where we use Lemmas A.3 and A.4 in the third and the fourth equalities respectively. The proof is complete. \square

LEMMA A.7.

$$\left\langle \begin{array}{c} i \uparrow \\ i+k \uparrow \\ i+k-1 \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-k \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = [j-1] \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \\ i+j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + [j-1] \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

for $j > k \geq 1$.

Proof. We first prove the case $k = 1$. We will show

$$\left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \\ i+j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + [j-1] \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

by induction on j . Note that Lemma A.3 is the case $i = 1$.

The left hand side becomes

$$\frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ i+1 \uparrow \\ i \uparrow \\ 1 \uparrow \\ j \uparrow \\ j-1 \uparrow \\ j \uparrow \\ i-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ i \uparrow \\ 1 \uparrow \\ j \uparrow \\ j+1 \uparrow \\ j \uparrow \\ i-1 \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n + \frac{[j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ i-1 \uparrow \\ j \uparrow \\ i+j-1 \uparrow \end{array} \right\rangle_n$$

$$\begin{aligned}
 &= \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow 1 \rightarrow \\ i-1 \uparrow \quad \downarrow j+1 \\ \leftarrow \quad \rightarrow \\ i+j \\ \downarrow \quad \uparrow \\ 1 \quad i+j-1 \end{array} \right\rangle_n + \frac{[j]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow 1 \rightarrow \\ i-1 \uparrow \quad \downarrow j+1 \\ \leftarrow \quad \rightarrow \\ i-2 \quad i+j-1 \\ \downarrow \quad \uparrow \\ 1 \quad i+j-1 \end{array} \right\rangle_n + \frac{[j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow i-1 \rightarrow \\ \downarrow \quad \uparrow \\ 1 \quad i+j-1 \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} i \uparrow \\ \leftarrow \quad \rightarrow \\ i+j \\ \downarrow \quad \uparrow \\ 1 \quad i+j-1 \end{array} \right\rangle_n + \frac{[j][i-1] + [j-i]}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow i-1 \rightarrow \\ \downarrow \quad \uparrow \\ 1 \quad i+j-1 \end{array} \right\rangle_n,
 \end{aligned}$$

which is equal to the right hand side. Here we use Lemma A.6 in the first equality and the inductive hypothesis in the second equality.

The proof for $k > 1$ is similar and omitted. \square

LEMMA A.8.

$$\left\langle \begin{array}{c} i \uparrow \\ \leftarrow j+k-i \rightarrow \\ j+k \uparrow \quad \downarrow i-k+1 \\ \leftarrow \quad \rightarrow \\ k \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n = \left\langle \begin{array}{c} i \uparrow \\ \leftarrow k \rightarrow \\ i-k \uparrow \quad \downarrow j+k+1 \\ \leftarrow \quad \rightarrow \\ j+k-i \quad i+1 \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} i \uparrow \\ \leftarrow k-1 \rightarrow \\ i-k+1 \uparrow \quad \downarrow j+k \\ \leftarrow \quad \rightarrow \\ j+k-i-1 \quad i+1 \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n$$

for $j \geq i \geq k \geq 1$.

Proof. For $k = 1$ we have

$$\begin{aligned}
 &\left\langle \begin{array}{c} i \uparrow \\ \leftarrow j-i+1 \rightarrow \\ j+1 \uparrow \quad \downarrow i \\ \leftarrow \quad \rightarrow \\ 1 \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow j-i+1 \rightarrow \\ j+1 \uparrow \quad \downarrow i-1 \\ \leftarrow \quad \rightarrow \\ 1 \quad i \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n = \frac{1}{[i]} \left\langle \begin{array}{c} i \uparrow \\ \leftarrow 1 \rightarrow \\ i-1 \uparrow \quad \downarrow j+1 \\ \leftarrow \quad \rightarrow \\ j-i+1 \quad i \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n \\
 &= \left\langle \begin{array}{c} i \uparrow \\ \leftarrow 1 \rightarrow \\ i-1 \uparrow \quad \downarrow j+2 \\ \leftarrow \quad \rightarrow \\ j-i+1 \quad i+1 \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n + \left\langle \begin{array}{c} i \uparrow \\ \leftarrow j-i \rightarrow \\ \downarrow \quad \uparrow \\ j \quad i+1 \end{array} \right\rangle_n,
 \end{aligned}$$

where we use Lemma A.7 in the last equality.

The proof for $k > 1$ is similar and left to the reader. \square

PROPOSITION A.9.

$$\sum_{k=0}^{i+1} (-1)^k \left\langle \begin{array}{c} i \uparrow \\ j+k \uparrow \\ j \uparrow \\ \leftarrow k \rightarrow \\ j+k-i \uparrow \\ i+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n = 0.$$

Proof. From Lemma A.8, we see that the left hand side equals

$$\begin{aligned} & \left\langle \begin{array}{c} i \uparrow \\ j-i \rightarrow \\ j \uparrow \\ i+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \\ \leftarrow k \rightarrow \\ j+k-i \uparrow \\ i+1 \uparrow \\ j+k+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n \\ & + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k+1 \uparrow \\ j \uparrow \\ \leftarrow k-1 \rightarrow \\ j+k-i-1 \uparrow \\ i+1 \uparrow \\ j+k \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n + (-1)^{i+1} \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \\ \leftarrow i+j+1 \rightarrow \\ i+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n \\ = & \left\langle \begin{array}{c} i \uparrow \\ j-i \rightarrow \\ j \uparrow \\ i+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n + \sum_{k=1}^i (-1)^k \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \\ \leftarrow k \rightarrow \\ j+k-i \uparrow \\ i+1 \uparrow \\ j+k+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n \\ & + \sum_{k=0}^{i-1} (-1)^{k+1} \left\langle \begin{array}{c} i \uparrow \\ i-k \uparrow \\ j \uparrow \\ \leftarrow k \rightarrow \\ j+k-i \uparrow \\ i+1 \uparrow \\ j+k+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n + (-1)^{i+1} \left\langle \begin{array}{c} i \uparrow \\ j \uparrow \\ \leftarrow i+j+1 \rightarrow \\ i+1 \uparrow \\ j+1 \uparrow \end{array} \right\rangle_n \end{aligned}$$

= 0, and the proof is complete. \square

More generally we have the following formula, which was suggested by J. Murakami. The proof is left to the reader.

PROPOSITION A.10.

$$\left\langle \begin{array}{c} i \uparrow \\ j+k \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{j+k-i} \\ \xleftarrow{k} \end{array} \begin{array}{c} j+l \uparrow \\ i-k+l \uparrow \\ i+l \uparrow \end{array} \right\rangle_n = \sum_{m=0}^i \begin{bmatrix} l \\ k-m \end{bmatrix} \left\langle \begin{array}{c} i \uparrow \\ i-m \uparrow \\ j \uparrow \end{array} \begin{array}{c} \xrightarrow{m} \\ \xrightarrow{j+m-i} \end{array} \begin{array}{c} j+l \uparrow \\ j+m+l \uparrow \\ i+l \uparrow \end{array} \right\rangle_n$$

for $j \geq i \geq k \geq 1$. Here $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ if $y < 0$ or $y > x$.