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The "if" direction did not need Φ to be irreducible. It can also be removed as a hypothesis in the "only if" direction by weakening the conclusion to Φ dividing a power of Θ.

Theorem 3 allowed Frobenius to establish a conjecture of Dedekind [10, p. 422], which said that the linear factors of Θ, monic in X_e, are related to the characters of the abelian group $G/[G,G]$. More precisely, Frobenius showed the linear factors of Θ, monic in X_e, are exactly the polynomials $\sum_g \chi(g)X_g$, where $\chi\colon G \to \mathbf{C}^\times$ is a character, and each such linear factor arises exactly once in the factorization of Θ. (Since we already showed such polynomials are factors, only the "if" direction of Theorem 3 is needed and therefore Lemma 1 is not required for this.) The reader is referred to the paper of Frobenius [22, Sect. 2] or Dickson [11, Sect. 6] for details of this argument.

It is of interest to see what is mentioned about the group determinant in Thomas Muir's *The Theory of Determinants in the Historical Order of Development*, which aimed to describe all developments in the subject up until 1900. In the preface to the final volume, Muir expresses the hope that "little matter of any serious importance has been passed over that was needed for this History." There are many references to the circulant, one to Dedekind's calculation of $\Theta(S_3)$, but there is no mention of any work on the group determinant by Frobenius. However, his List of Writings in the 1907 *Quart. J. Pure Appl. Math.* shows he was aware of such papers.

5. Factoring the Group Determinant by Representation Theory

We now use representation theory to completely factor the group determinant. As in the second proof of Theorem 2, let's compute the matrix for left multiplication in $\mathbf{C}[G]$ by an element $\sum a_g g$, with respect to the basis G of $\mathbf{C}[G]$. Since

$$\Big(\sum_g a_g g\Big)h = \sum_g a_{gh^{-1}} g,$$

the matrix for left multiplication by $\sum a_g g$ is $(a_{gh^{-1}})$. Hence

$$\det(a_{gh^{-1}}) = \mathrm{N}_{\mathbf{C}[G]/\mathbf{C}}\Big(\sum_g a_g g\Big).$$

Since $\mathbf{C}[G]$ decomposes into a product of matrix algebras, this norm will decompose into a product of determinants. More specifically, let $\{(\rho, V_\rho)\}$ be a full set of mutually nonisomorphic irreducible representations of G (over the complex numbers). Then the map

$$\mathbf{C}[G] \to \prod_{\rho \text{ irred}} \mathrm{End}_{\mathbf{C}}(V_\rho),$$

given by

$$\sum_{g\in G} a_g g \mapsto \Big(\sum_{g\in G} a_g \rho(g)\Big)_{\rho \text{ irred}},$$

is an isomorphism of $\mathbf{C}$-algebras. Thus

$$\begin{aligned} \mathrm{N}_{\mathbf{C}[G]/\mathbf{C}}\Big(\sum_g a_g g\Big) &= \prod_{\rho \text{ irred}} \mathrm{N}_{\mathrm{End}_{\mathbf{C}}(V_\rho)/\mathbf{C}}\Big(\sum_g a_g \rho(g)\Big) \\ &= \prod_{\rho \text{ irred}} \det\Big(\sum_g a_g \rho(g)\Big)^{\deg(\rho)}. \end{aligned}$$

This last equation arises from the fact that in the endomorphism ring $\mathrm{End}(V)$ of an m-dimensional vector space V, left multiplication by an element is a linear map $\mathrm{End}(V) \to \mathrm{End}(V)$ whose determinant is equal to the m-th power of the usual determinant of the element. Therefore

$$\Theta(G) = \det(X_{gh^{-1}}) = \prod_{\rho \text{ irred}} \det\Big(\sum_g X_g \rho(g)\Big)^{\deg(\rho)}.$$

Note $\det(\sum_g X_g \rho(g))$ is a homogeneous polynomial of degree $\deg(\rho)$, monic in X_e.

We now show that the irreducible factors of $\Theta(G)$ (which are monic in X_e) can be put in a one-to-one correspondence with the irreducible representations of G by proving

THEOREM 4. *For an irreducible complex representation ρ of G,*

(i) *the polynomial* $\det(\sum_g X_g \rho(g))$ *is irreducible and*

(ii) *ρ is determined by* $\det(\sum_g X_g \rho(g))$.

We begin with a lemma originally due to Burnside [4].

LEMMA 2. *If (ρ, V) is an irreducible representation of G, then the $\mathbf{C}$-algebra map* $\mathbf{C}[G] \to \mathrm{End}_{\mathbf{C}}(V)$ *given by* $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \rho(g)$ *is onto. That is, the transformations $\rho(g)$ linearly span* $\mathrm{End}_{\mathbf{C}}(V)$.

Proof. This map is basically a projection of $\mathbf{C}[G]$ onto one of its simple $\mathbf{C}[G]$-submodules, so it is onto. Alternatively, for a proof that works for representations over any algebraically closed field, even one with characteristic dividing the size of G, see [33]. □

LEMMA 3. *Let $\rho: G \to \mathrm{GL}_d(\mathbf{C})$ be a representation. Write*

$$\sum_{g\in G} X_g \rho(g) = (L_{ij}),$$

where the L_{ij}'s are linear polynomials in the X_g's. If ρ is irreducible then the L_{ij}'s are linearly independent over $\mathbf{C}$.

Proof. By Lemma 2, any set of d^2 complex numbers (z_{ij}) arises as $\sum a_g \rho(g) = (L_{ij}(a_g))$ for some vector (a_g) in $\mathbf{C}^n$. So

$$\begin{aligned}\sum c_{ij} L_{ij} = 0 \text{ in } \mathbf{C}[X_g] &\Rightarrow \sum c_{ij} L_{ij}(a_g) = 0 \text{ for all } (a_g) \in \mathbf{C}^n \\ &\Rightarrow \sum c_{ij} z_{ij} = 0 \text{ for all } (z_{ij}) \in \mathbf{C}^{d^2} \\ &\Rightarrow \text{all } c_{ij} = 0. \qquad \square\end{aligned}$$

Proof of Theorem 4. (i) By Lemma 3, choose $n - \deg(\rho)^2$ homogeneous linear polynomials L_k such that $\{L_{ij}, L_k\}$ is a basis of the homogeneous linear polynomials in $\mathbf{C}[X_g]$. Then we can move between the sets $\{X_g\}$ and $\{L_{ij}, L_k\}$ by a linear change of variables. This gives a $\mathbf{C}$-algebra automorphism of $\mathbf{C}[X_g]$, so the set $\{L_{ij}, L_k\}$ consists of algebraically independent elements over $\mathbf{C}$. In particular,

$$\det\Big(\sum_{g\in G} X_g \rho(g)\Big) = \det(L_{ij})$$

is the determinant of a matrix whose entries are algebraically independent. It is a standard fact (see [36, p. 96] for an elementary proof) that such a determinant is irreducible in $\mathbf{C}[L_{ij}]$, so it is also irreducible if we append the extra algebraically independent variables $\{L_k\}$ to the ring, so this polynomial is irreducible in $\mathbf{C}[L_{ij}, L_k] = \mathbf{C}[X_g]$.

(ii) We need to show that ρ is determined by $\det(\sum X_g \rho(g))$. It is enough to show the corresponding character χ_ρ is determined, and that is what we will do.

The number $\chi_\rho(e)$ is the degree of the homogeneous polynomial $\det(\sum X_g \rho(g))$. For $h \neq e$, we will recover $\chi_\rho(h)$ as the coefficient of $X_e^{\deg(\rho)-1} X_h$. To see this, we ignore all variables besides X_e and X_h by setting X_g equal to 0 for $g \neq e, h$. Then our polynomial becomes $\det(X_e I + X_h \rho(h))$. We want to know the coefficient of $X_e^{\deg(\rho)-1} X_h$ in this polynomial. For any matrix A, the polynomial $\det(TI + A)$ in the variable T has second leading coefficient $\mathrm{Tr}(A)$. Apply this to $A = X_h \rho(h)$, whose trace is $\chi_\rho(h) X_h$. $\square$

Let's work through the proof of Theorem 4(i) in a case we've already seen, $G = S_3$. Recall

$$\pi_1 = (1),\ \pi_2 = (123),\ \pi_3 = (132),\ \pi_4 = (23),\ \pi_5 = (13), \pi_6 = (12).$$

Let $\rho\colon S_3 \to \mathrm{GL}(V)$ be the irreducible 2-dimensional representation on

$$V = \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\},$$

given by permutation of the cooordinates. Using $(1,0,-1), (0,1,-1)$ as an ordered basis of V, we get the matrix realizations

$$[\rho(\pi_1)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\rho(\pi_2)] = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad [\rho(\pi_3)] = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$
$$[\rho(\pi_4)] = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad [\rho(\pi_5)] = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad [\rho(\pi_6)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore

$$(5.1) \qquad \sum_{i=1}^{6} X_i[\rho(\pi_i)] = \begin{pmatrix} X_1 - X_2 + X_4 - X_5 & -X_2 + X_3 - X_5 + X_6 \\ X_2 - X_3 - X_4 + X_6 & X_1 - X_3 - X_4 + X_5 \end{pmatrix},$$

which tells us what the L_{ij} in Lemma 3 are, $1 \leq i, j \leq 2$. Taking the determinant of the right hand side of (5.1) gives an expression $ad - bc$ for the factor Φ_3 of $\Theta(S_3)$ where a, b, c, d are linear polynomials with integer coefficients (such an expression was given by Dickson in [14, Eq. 2]). In the expression of Dedekind's for Φ_3 which we saw earlier, a, b, c, and d had coefficients involving cube roots of unity. The fact that we can get integer coefficients is related to the 2-dimensional irreducible representation of S_3 being realizable in $\mathrm{GL}_2(\mathbf{Z})$. In general, the irreducible factors of $\Theta(S_n)$ have integer coefficients since all irreducible representations of S_n are defined over the rational numbers.

As a basis of the linear forms in $\mathbf{C}[X_i]$ we use the L_{ij} and matrix entries of all $\sum X_i\rho'(\pi_i)$ where ρ' runs over irreducible representations of S_3 not isomorphic to ρ. These are the trivial and sign representations, which yield $L_1 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$ and $L_2 = X_1 + X_2 + X_3 - X_4 - X_5 - X_6$, so we can also use $X_1 + X_2 + X_3$ and $X_4 + X_5 + X_6$. These are essentially elements Dedekind came across when factoring $\Theta(S_3)$ into linear factors in some hypercomplex number system. Compare this means of manufacturing $\pi_1 + \pi_2 + \pi_3$ and $\pi_4 + \pi_5 + \pi_6$ with Dedekind's calculation.

As an illustration of the proof of Theorem 4(ii), the quadratic irreducible factor of $\Theta(S_3)$ corresponds to the irreducible 2-dimensional representation of S_3, and its coefficients of X_1X_i for $2 \le i \le 6$ (some of which are zero) coincide with the character values at π_i.

The proof given for Frobenius' theorem on the factorization of $\Theta(G)$ can be adapted to show that for any finite-dimensional complex representation ρ of G, the *determinant attached to* ρ, namely

$$\Theta_\rho(G) = \det\Big(\sum_{g\in G} X_g\rho(g)\Big),$$

decomposes into homogeneous irreducible factors (monic in X_e) in accordance with the decomposition of ρ into irreducible representations. Frobenius' theorem on the group determinant involves the regular representation.

In Frobenius' initial work on the group determinant, he felt the most remarkable (and difficult to prove) feature of the factorization was that the degree of each irreducible factor coincides with its multiplicity as a factor. We recognize this feature as a familiar statement about the multiplicity of irreducible representations in the regular representation.

Since every factor (monic in X_e) of the group determinant has the form $\det(\sum_g X_g\rho(g))$ for some representation ρ, the "if" direction of Theorem 3 gets a second proof from the definition of a representation and the multiplicativity of determinants.

According to Hawkins [26, 27], Frobenius' original approach to characters of G (which is not the first one that appeared in print) was as follows. Let Φ be an irreducible factor of $\Theta(G)$ which is monic in X_e and of degree d. Define the associated character χ by letting $\chi(g)$ be the coefficient of X_e^{d-1} in $\partial\Phi/\partial X_g$. This is equivalent to the description we gave in the proof of Theorem 4(ii), except that we speak of the character attached to an irreducible representation of G while Frobenius (at first) spoke of the character attached to an irreducible factor of the group determinant of G.

Here is another point of view that Frobenius had on characters. Let Φ be an irreducible factor of the group determinant of G, monic in X_e and of degree d. We regard Φ as a function $\mathbf{C}[G] \to \mathbf{C}$ by $\sum a_g g \mapsto \Phi(a_g)$. Let $x = \sum a_g g \in \mathbf{C}[G]$. For a variable u, set

$$\Phi(x + ue) = u^d + C_1u^{d-1} + \cdots + C_d, \tag{5.2}$$

where C_i is a polynomial function of the a_g's which is homogeneous of degree i. In particular, C_1 is a linear homogeneous polynomial of the a_g's. Frobenius observed in [22, p. 1360] that its coefficients are the values of the character χ corresponding to Φ: $C_1 = \sum_g \chi(g)a_g$. Since (5.2) is essentially a characteristic polynomial, so C_1 is basically a trace, the connection Frobenius eventually found between characters and traces is not surprising.

In [22], Frobenius explicitly showed how *all* the coefficients of an irreducible factor of the group determinant can be expressed explicitly in terms of its corresponding character. We will show more generally that for any (complex) representation ρ of G, irreducible or not, the coefficients of $\det(\sum X_g\rho(g))$ can be expressed in terms of χ_ρ. Our discussion is based on the matrix formula (5.3) below, which we now explain.

For $N \geq 1$ and $\sigma \in S_N$ consisting of disjoint cycles of length $N_1, \ldots, N_r$, define a trace map $\mathrm{Tr}_\sigma \colon \mathrm{M}_d(\mathbf{C}) \to \mathbf{C}$ by $\mathrm{Tr}_\sigma(A) = \mathrm{Tr}(A^{N_1}) \cdot \ldots \cdot \mathrm{Tr}(A^{N_r})$. For example, $\mathrm{Tr}_{(1)(2)\ldots(N)}(A) = (\mathrm{Tr}\,A)^N$, $\mathrm{Tr}_{(1,\ldots,N)}(A) = \mathrm{Tr}(A^N)$, and $\mathrm{Tr}_\sigma(I_d) = d^r$. If σ and τ are conjugate in S_N, they have the same cycle structure (and vice versa), so $\mathrm{Tr}_\sigma = \mathrm{Tr}_\tau$. Note Tr_σ is typically not linear.

For our application, we set $N = d$. We will prove that for $A \in \mathrm{M}_d(\mathbf{C})$,

$$\det(A) = \frac{1}{d!} \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma)\,\mathrm{Tr}_\sigma(A). \tag{5.3}$$

A formula equivalent to (5.3) was used by Frobenius in [22, Sect. 3, Eq. 8].

For example, when $d = 2$ let A have eigenvalues λ and μ. The right hand side is

$$\frac{1}{2}\left((\mathrm{Tr}\,A)^2 - \mathrm{Tr}(A^2)\right) = \frac{1}{2}\left((\lambda+\mu)^2 - (\lambda^2+\mu^2)\right) = \lambda\mu = \det(A).$$

To prove (5.3), let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A, repeated with multiplicity. For $k \geq 1$, let $s_k = \lambda_1^k + \cdots + \lambda_d^k$.

If σ has m_1 1-cycles, m_2 2-cycles, and so on, then $m_1 + 2m_2 + \cdots + dm_d = d$ and $\mathrm{sgn}(\sigma) = \prod_k \left((-1)^{k-1}\right)^{m_k}$. Since $\sum k m_k = d$, $\mathrm{sgn}(\sigma) = (-1)^{d - \sum_k m_k}$. Also, $\mathrm{Tr}_\sigma(A) = s_1^{m_1} s_2^{m_2} \cdot \ldots \cdot s_d^{m_d}$. Therefore

$$\mathrm{sgn}(\sigma)\,\mathrm{Tr}_\sigma(A) = (-1)^d \prod_{k=1}^{d} (-1)^{m_k} s_k^{m_k}.$$

If σ and τ have the same cycle structure, $\mathrm{sgn}(\sigma)\,\mathrm{Tr}_\sigma(A) = \mathrm{sgn}(\tau)\,\mathrm{Tr}_\tau(A)$. For our evaluation of

$$\frac{1}{d!} \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma)\,\mathrm{Tr}_\sigma(A),$$

we want to collect all the terms corresponding to permutations with the same cycle structure. The permutations in S_d having a cycle structure with m_1 1-cycles, m_2 2-cycles, and so on form a conjugacy class whose size is $d!/\prod_{k=1}^{d} k^{m_k} \cdot m_k!$. Thus

$$\frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\operatorname{Tr}_\sigma(A)=\frac{1}{d!}\sum_{\substack{m_1,m_2,\cdots\geq 0\\ m_1+2m_2+\cdots=d}}(-1)^d d!\prod_{k=1}^{d}\frac{(-1)^{m_k}s_k^{m_k}}{k^{m_k}m_k!}$$

$$=(-1)^d\sum_{\substack{m_1,m_2,\cdots\geq 0\\ m_1+2m_2+\cdots=d}}\prod_{k=1}^{d}\frac{(-1)^{m_k}s_k^{m_k}}{k^{m_k}m_k!}.$$

We want to show this equals $\lambda_1\cdot\ldots\cdot\lambda_d$. To do this, we use generating functions:

$$\sum_{i\geq 0}\Big(\sum_{\substack{m_1,m_2,\cdots\geq 0\\ m_1+2m_2+\cdots=i}}\prod_{k=1}^{d}\frac{(-1)^{m_k}s_k^{m_k}}{k^{m_k}m_k!}\Big)t^i=\sum_{i\geq 0}\sum_{\substack{m_1,m_2,\cdots\geq 0\\ m_1+2m_2+\cdots=i}}\prod_{k=1}^{d}\frac{(-1)^{m_k}(s_kt^k)^{m_k}}{k^{m_k}m_k!}$$

$$=\prod_{k=1}^{d}\sum_{m_k\geq 0}\Big(\frac{-s_kt^k}{k}\Big)^{m_k}\frac{1}{m_k!}=\prod_{k=1}^{d}e^{-s_kt^k/k}$$

$$=\exp\Big(-\sum_{k=1}^{d}\frac{s_kt^k}{k}\Big)=\exp\Big(-\sum_{j=1}^{d}\sum_{k=1}^{d}\frac{\lambda_j^kt^k}{k}\Big)$$

$$=\prod_{j=1}^{d}\exp\Big(-\sum_{k=1}^{d}\lambda_j^kt^k/k\Big)\equiv\prod_{j=1}^{d}\exp(\log(1-\lambda_jt))\mod t^{d+1}$$

$$\equiv\prod_{j=1}^{d}(1-\lambda_jt)\mod t^{d+1}.$$

The coefficient of t^d here is $(-1)^d\lambda_1\cdot\ldots\cdot\lambda_d$, as desired.

More generally, for $N\geq 1$ and $A\in M_d(\mathbf{C})$, the coefficient of t^N in $\prod_{j=1}^d(1-\lambda_jt)$ is $(-1)^N\operatorname{Tr}(\bigwedge^N A)$, so by an argument similar to the one above,

$$\operatorname{Tr}\Big(\bigwedge^N A\Big)=(-1)^N\sum_{\substack{m_1,m_2,\cdots\geq 0\\ m_1+2m_2+\cdots=N}}\prod_{k=1}^{N}\frac{(-1)^{m_k}s_k^{m_k}}{k^{m_k}m_k!}=\frac{1}{N!}\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\operatorname{Tr}_\sigma(A).$$

It is interesting to write (5.3) using the classical definition of the determinant of the $d\times d$ matrix (a_{ij}):

$$\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdot\ldots\cdot a_{d\sigma(d)}=\frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\operatorname{Tr}_\sigma((a_{ij})).$$

Although these sums are both taken over S_d, the addends corresponding to the same permutation σ are typically not equal. For instance, for a diagonal matrix the left hand side has only one nonzero term while the right hand side has many nonzero terms.

Let's apply (5.3) to representation theory. It says that for a d-dimensional representation ρ of G,

$$\det\Big(\sum_{g\in G} X_g\rho(g)\Big) = \frac{1}{d!}\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\operatorname{Tr}_\sigma\Big(\sum_{g\in G} X_g\,\rho(g)\Big)$$

$$= (-1)^d \sum_{\substack{m_1,m_2,\dots\geq 0\\ m_1+2m_2+\cdots=d}} \prod_{k=1}^{d} \frac{(-1)^{m_k}}{k^{m_k} m_k!}\left(\operatorname{Tr}\Big(\Big(\sum_g \rho(g)X_g\Big)^k\Big)\right)^{m_k}.$$

which equals

$$(-1)^d \sum_{\substack{m_1,m_2,\dots\geq 0\\ m_1+2m_2+\cdots=d}} \prod_{k=1}^{d} \frac{(-1)^{m_k}}{k^{m_k} m_k!}\Big(\sum_{(g_1,\dots,g_k)\in G^k} \chi_\rho(g_1\cdot\ldots\cdot g_k)X_{g_1}\cdot\ldots\cdot X_{g_k}\Big)^{m_k}.$$

So all coefficients can be expressed in terms of χ_ρ. For the connection between the coefficients and the higher characters of ρ, see Johnson [30, p. 301].

In particular, if ρ is 1-dimensional then $\det\Big(\sum X_g\rho(g)\Big) = \sum \chi_\rho(g)X_g$. For 2-dimensional ρ,

$$\det\Big(\sum X_g\rho(g)\Big) = \frac{1}{2}\Big(\sum_{g\in G}\chi_\rho(g)X_g\Big)^2 - \frac{1}{2}\sum_{(g,h)\in G^2}\chi_\rho(gh)X_gX_h$$

$$= \frac{1}{2}\sum_{(g,h)\in G^2}(\chi_\rho(g)\chi_\rho(h) - \chi_\rho(gh))X_gX_h$$

$$= \frac{1}{2}\sum_{g}(\chi_\rho(g)^2 - \chi_\rho(g^2))X_g^2$$

$$+ \sum_{\{g,h\}\ \text{unequal}}(\chi_\rho(g)\chi_\rho(h) - \chi_\rho(gh))\,X_gX_h\,.$$

To conclude this section, let's use the point of view developed here to factor the group determinant of D_8, the group of symmetries of the square (also denoted by some as D_4). We index the elements of D_8 as

$$g_1 = 1, \qquad g_2 = (13)(24), \qquad g_3 = (1234), \qquad g_4 = (1432),$$
$$g_5 = (13), \qquad g_6 = (24), \qquad g_7 = (12)(34), \qquad g_8 = (14)(23).$$

The conjugacy classes are

$$c_1 = \{1\}\,,\ c_2 = \{g_2\}\,,\ c_3 = \{g_3, g_4\}\,,\ c_4 = \{g_5, g_6\}\,,\ c_5 = \{g_7, g_8\}\,.$$

The character table of D_8 is

	c_1	c_2	c_3	c_4	c_5
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

Therefore $\Theta(D_8) = \Phi_1\,\Phi_2\,\Phi_3\,\Phi_4\,\Phi_5^2$, where

$$\begin{aligned}
\Phi_1 &= X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8\,,\\
\Phi_2 &= X_1 + X_2 + X_3 + X_4 - X_5 - X_6 - X_7 - X_8\,,\\
\Phi_3 &= X_1 + X_2 - X_3 - X_4 + X_5 + X_6 - X_7 - X_8\,,\\
\Phi_4 &= X_1 + X_2 - X_3 - X_4 - X_5 - X_6 + X_7 + X_8\,,\\
\Phi_5 &= \det\Bigl(\sum X_g \rho(g)\Bigr)\,.
\end{aligned}$$

where ρ is the 2-dimensional irreducible representation of D_8. So

$$\begin{aligned}
\Phi_5 &= \sum_g \frac{1}{2}(\chi_5(g)^2 - \chi_5(g^2))X_g^2 + \sum_{\{g,h\}\ \text{unequal}} (\chi_5(g)\chi_5(h) - \chi_5(gh))\,X_g X_h\\
&= X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 - X_6^2 - X_7^2 - X_8^2\\
&\quad - 2X_1X_2 - 2X_3X_4 + 2X_5X_6 + 2X_7X_8\,.
\end{aligned}$$

Although Q_8 and D_8 have identical character tables, and all coefficients of an irreducible factor of the group determinant are determined by the corresponding character, the quadratic irreducible factors of $\Theta(Q_8)$ and $\Theta(D_8)$ are different. This illustrates that the determination of all coefficients of a factor from its character depends on the character as a function on group elements, not only on conjugacy classes.