Zeitschrift: L'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DIOPHANTINE EQUATION INVOLVING FIFTH POWERS
Autor: Choudhry, Ajai
DOI: https://doi.org/10.5169/seals-63895

L'Enseignement Mathématique, t. 44 (1998), p. 53–55

A DIOPHANTINE EQUATION INVOLVING FIFTH POWERS

by Ajai CHOUDHRY

ABSTRACT. The paper provides a parametric solution of the diophantine equation $aX_1^5+bX_2^5+cX_3^5+dX_4^5 = aY_1^5+bY_2^5+cY_3^5+dY_4^5$, where a,b,c,d are arbitrary non-zero integers.

While several parametric solutions of the diophantine equation

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 = Y_1^5 + Y_2^5 + Y_3^5 + Y_4^5$$

are known [1,2,3], the diophantine equation

$$aX_1^5 + bX_2^5 + cX_3^5 + dX_4^5 = aY_1^5 + bY_2^5 + cY_3^5 + dY_4^5 \tag{1}$$

has not been considered earlier. In this paper we give a parametric solution of (1) when a,b,c,d are arbitrary non-zero integers.

To solve (1), we write

$$\left\{\begin{array}{ll} X_1 = m_1p_1u + m_2v\alpha, & X_2 = m_1q_1u + m_2v\beta, \\ X_3 = n_1p_2u + n_2v\alpha, & X_4 = n_1q_2u + n_2v\beta, \\ Y_1 = m_1r_1u + m_2v\alpha, & Y_2 = m_2v\beta, \\ Y_3 = n_1r_2u + n_2v\alpha, & Y_4 = n_2v\beta, \end{array}\right. \tag{2}$$

where $m_1, n_1, p_1, q_1, r_1, m_2, n_2, p_2, q_2, r_2, u, v, \alpha, \beta$ are arbitrary. Substituting these values in (1), we get an equation which may be written as

$$\sum_{i=1}^{5} \binom{5}{i} u^i v^{5-i} \Big[m_1^i m_2^{5-i} \left\{ a(p_1^i - r_1^i)\alpha^{5-i} + bq_1^i\beta^{5-i} \right\} + n_1^i n_2^{5-i} \left\{ c(p_2^i - r_2^i)\alpha^{5-i} + dq_2^i\beta^{5-i} \right\} \Big] = 0 . \tag{3}$$

We now choose $p_1, q_1, r_1, p_2, q_2, r_2$ as follows:

$$(4)\qquad \begin{cases} p_1 = a\alpha^5 - b\beta^5, & q_1 = 2a\alpha^4\beta, \quad r_1 = a\alpha^5 + b\beta^5, \\ p_2 = -c\alpha^5 + d\beta^5, & q_2 = -2c\alpha^4\beta, \quad r_2 = -c\alpha^5 - d\beta^5. \end{cases}$$

With these values of $p_1, q_1, r_1, p_2, q_2, r_2$ (and m_1, n_1, m_2, n_2 arbitrary), we find that the coefficients of uv^4 and u^2v^3 in equation (3) become zero. The coefficient of u^3v^2 in (3) will also vanish if

$$(5)\qquad abm_1^3m_2^2(a^2\alpha^{10} - b^2\beta^{10}) - cdn_1^3n_2^2(c^2\alpha^{10} - d^2\beta^{10}) = 0.$$

We accordingly choose

$$(6)\qquad \begin{cases} m_1 = n_2, \quad n_1 = m_2, \\ m_2 = ab(a^2\alpha^{10} - b^2\beta^{10}), \quad n_2 = cd(c^2\alpha^{10} - d^2\beta^{10}), \end{cases}$$

so that (5) is satisfied. Now, equation (3) has only the terms involving u^4v and u^5 and it is readily solved to give the following solution for u, v:

$$(7)\qquad \begin{cases} u = -20a^2b\alpha^6m_1^4m_2(a^2\alpha^{10} - b^2\beta^{10}) \\ \qquad\qquad - 20c^2d\alpha^6n_1^4n_2(c^2\alpha^{10} - d^2\beta^{10}), \\ v = abm_1^5(11a^4\alpha^{20} - 10a^2b^2\alpha^{10}\beta^{10} - b^4\beta^{20}) \\ \qquad\qquad - cdn_1^5(11c^4\alpha^{20} - 10c^2d^2\alpha^{10}\beta^{10} - d^4\beta^{20}). \end{cases}$$

Thus, a parametric solution of equation (1) in terms of the parameters α and β is given by (2), where m_1, n_1, m_2, n_2 are defined by (6), $p_1, q_1, r_1, p_2, q_2, r_2$ by (4), and u, v by (7).

As a numerical example, when $a = 1$, $b = 2$, $c = 3$, $d = 6$, taking $\alpha = 1$, $\beta = 2$, we get the following solution of (1):

$$(X_1, X_2, X_3, X_4) = (1955587, 2963474, 121184667, 242404434),$$
$$(Y_1, Y_2, Y_3, Y_4) = (1022467, 2992634, 121219227, 242403354).$$

REFERENCES

[1] CHOUDHRY, A. The diophantine system $\sum_{i=1}^4 x_i^r = \sum_{i=1}^4 y_i^r$, $r = 1, 3, 5$. *Bull. Calcutta Math. Soc. 83* (1991), 85–86.

[2] LANDER, L. J. Geometric aspects of diophantine equations involving equal sums of like powers. *Amer. Math. Monthly 75* (1968), 1061–1073.

[3] LANDER, L. J., T. R. PARKIN and J. L. SELFRIDGE. A survey of equal sums of like powers. *Math. Comp. 21* (1967), 446–459.

(Reçu le 5 décembre 1997)

Ajai Choudhry
Embassy of India
Kantari Street, Sahmarani Building
P.O. Box No. 113-5240
Beirut
Lebanon
e-mail: indembei@dm.net.lb