Appendix : More on some proofs of full reducibility

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However, $SL_2(k)$ does not leave any non-zero element of V_2 stable, so V_2 is not fully reducible.

This does not rule out a positive answer to problem I, but if so, another approach had to be devised. D. Mumford proposed a weaker notion than full reducibility, now called geometric reductivity: if C is an invariant one-dimensional subspace, there exists a homogeneous G-invariant hypersurface not containing C (in the case of full reducibility it could be a hyperplane). Then Nagata showed that this condition indeed implies the finite generation of the algebra of invariants. Later geometric reductivity was proved by C.S. Seshadri for $SL_2(k)$ and by W. Haboush in general.

Even over $\mathbf{C}$, the problems of full reducibility and of the determination of irreducible representations resurfaced not for $SL_2(\mathbf{C})$, but for its generalization as a Kac-Moody Lie algebra, or for the deformation of its Lie algebra as a "quantum group". This has led to further problems and to more contacts with mathematical physics.

Appendix: More on some proofs of full reducibility

We give here more technical details on the proofs of full reducibility for $\mathfrak{sl}_2(\mathbf{C})$ or $SL_2(\mathbf{C})$ due to Cartan, Fano and Casimir, assuming some familiarity with Lie algebras and algebraic geometry. We let $\mathfrak{g}$ stand for $\mathfrak{sl}_2(\mathbf{C})$.

12. Lie algebra proof:

12.1. Let

(1) $$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

be the familiar basis of $\mathfrak{g}$. It satisfies the relations

(2) $$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = -h.$$

The elements h, e, f define one-parameter subgroups ($t \in \mathbf{R}$)

$$e^{th} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad e^{te} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad e^{tf} = \begin{pmatrix} 1 & 0 \\ -t & 0 \end{pmatrix}.$$

By letting them act on functions of x, y and taking the derivatives for $t = 0$, we get expressions of h, e, f as differential operators, namely

(3) $$h = x\,.\,\partial_x - y\,.\,\partial_y, \quad e = x\,.\,\partial_y, \quad f = -y\,.\,\partial_x\,.$$

Let E be a representation space for $\mathfrak{g}$ and E_c ($c \in \mathbf{C}$) the eigenspace for h with eigenvalue c. Then (2) implies

$$(4) \qquad e\,.\,E_c \subset E_{c+2} \qquad f\,.\,E_c \subset E_{c-2}\,.$$

More generally, if $(h - c\,.\,I)^q\,.\,v = 0$ for some $q \geq 1$, then

$$(5) \qquad (h-(c+2)\,.\,I)^q\,.\,e\,.\,v = 0 = (h-(c-2)\,.\,I)^q\,.\,f\,.\,v = 0\,.$$

12.2. We now consider V_m. It has a basis $x^{m-i}\,.\,y^i$ ($i = 0, \ldots, m$) and $x^{m-i}\,.\,y^i$ is an eigenvector for h, with eigenvalue $m - 2i$. Let

$$(1) \qquad v_{m-2i} = \binom{m}{i}\, x^{m-i}\,.\,y^i \qquad (i = 0, \ldots, m)\,.$$

The v_{m-2i} form a basis of V_m and we have:

$$(2) \qquad h\,.\,v_{m-2i} = (m-2i)v_{m-2i} \qquad (i = 0, \ldots, m)\,.$$

A simple computation, using 12.1(2), (3), yields

$$(3) \qquad f\,.\,v_{m-2i} = -(i+1)\,v_{m-2i-2}$$

$$(4) \qquad e\,.\,v_{m-2i} = (m-i+1)\,v_{m-2i+2} \qquad (i = 0, \ldots, m)\,,$$

with the understanding that

$$(5) \qquad v_{m+2} = v_{-m-2} = 0\,.$$

(3) and (4) imply

$$(6) \qquad f\,.\,e\,.\,v_{m-2i} = -i(m-i+1)v_{m-2i}$$

$$(7) \qquad e\,.\,f\,.\,v_{m-2i} = (i+1)(m-i)v_{m-2i}\,.$$

REMARKS. (a) The eigenvalues of h in V_m are integers. By consideration of a Jordan-Hölder series, it follows that this is true for any finite dimensional representation.

(b) In $\mathbf{P}(V_m)$ the rational normal curve occurring in Lie's description of the irreducible projective representations of $\mathrm{SL}_2(\mathbf{C})$ (see §2) is the orbit of the point representing the line spanned by x^m. This is also the unique fixed point in $\mathbf{P}(V_m)$ of the group U generated by e, i.e. the group of upper triangular unipotent (eigenvalues equal to one) matrices. It is therefore also the locus of the fixed points of the conjugates of U in $\mathrm{SL}_2(\mathbf{C})$, and each such conjugate has a unique fixed point in $\mathbf{P}(V_m)$.

12.3. Note that 12.2(5) is a consequence of 12.2(4) and of the commutation relations 12.1(1). A similar argument shows more generally that if E is a representation of $\mathfrak{g}$ and $v \in E$ satisfies the conditions

(1) $$e \,.\, v = 0, \quad h \,.\, v = c \,.\, v \qquad (c \in \mathbf{C}),$$

then the elements $f^i \,.\, v$ $(i \geq 0)$ span a finite dimensional $\mathfrak{g}$-submodule F. In particular $\mathbf{C} \,.\, v$ is the eigenspace with eigenvalue c and all other eigenvalues of h in F are of the form $c - q$ $(q \in \mathbf{N},\, q \geq 1)$.

12.4. *First proof of full reducibility.* We use 12.3, which is contained in [Cr1] and the two remarks a) and b) of §3. This reduces the proof of full reducibility of a $\mathfrak{g}$-module E to the case of a short exact sequence

(1) $$0 \to V_m \to E \xrightarrow{\pi} V_n \to 0 \qquad (m \leq n).$$

Let $m < n$. Then h has an eigenvector $v \in E$ with eigenvalue n, which does not belong to V_m. It is annihilated by e, since there are no weights $> n$ in V_m or V_n, hence in E. By 12.3, it generates a $\mathfrak{g}$-submodule distinct from V_m, which must therefore be a $\mathfrak{g}$-invariant complement to V_m.

Let now $m = n$. Let $\{v_{m-2i}\}$ $(i = 0, \ldots, m)$ be the basis of V_m, viewed as subspace of E, constructed in 12.2. Let v'_m be a vector which maps under π onto the similar basis element of the quotient and let $v'_{m-2i} = \binom{m}{i} \,.\, v'_m$. Then the v'_{m-2i} project onto the basis of E/V_m defined in 12.2. There exists $a \in \mathbf{C}$ such that

(2) $$h \,.\, v'_m = m \,.\, v'_m + a \,.\, v_m \,.$$

We claim it suffices to show that $a = 0$. Indeed, in that case, 12.3 again implies that v'_m generates a $\mathfrak{g}$-submodule distinct from V_m, hence a supplement to V_m.

There remains to prove that $a = 0$. We claim first

(3) $$h \,.\, v'_{m-2i} = (m - 2i)v'_{m-2i} + a \,.\, v_{m-2i} \qquad (i = 0, \ldots, m).$$

For $i = 0$, this is (2). Assuming it is proved for i, we obtain (3) for $i + 1$ by applying f to both sides and using 12.1(2), 12.2(3).

For $i \geq 1$, we have, by 12.1(2) and 12.2(3)

(4) $$i \,.\, e \,.\, v'_{m-2i} = -e \,.\, f \,.\, v'_{m-2i+2} = -f \,.\, e \,.\, v'_{m-2i+2} + h \,.\, v'_{m-2i+2} \,.$$

By (3) and 12.2(6), this yields

(5) $$i \,.\, e \,.\, v'_{m-2i} = i(m - i + 1) \,.\, v'_{m-2i+2} + a \,.\, v_{m-2i+2} \,.$$

If we apply (5) for $i = m + 1$, we get $a \,.\, v_{-m} = 0$, hence $a = 0$.

REMARK. This last computation is contained in [CW] and also, unknown to the authors, in [Cr1]. As we saw, the proof for $m < n$ reduces immediately to 12.3, and by b) in §3 it suffices to consider that case when $m \neq n$. A direct computation along the lines of the previous proof is longer if $m > n$ (see 12.5). Cartan performs it even for a Jordan-Hölder series of any length, which leads to a rather complicated argument. By using his operator, Casimir did not have to make any distinction between the cases $m < n$ and $m > n$.

12.5. To give a better idea of Cartan's proof, we discuss the case $m > n$ directly, without reducing to $m < n$.

We let v'_n and v'_{n-2i} $(i \geq 0)$ be as before. Note first that if n and m have different parities, then V_n and V_m have no common eigenvalue for h. In particular h has no element of weight $n+2$ in E and the eigenspace for n is one-dimensional, hence spanned by v'_n. Again, by 12.3, v'_n generates a complementary $\mathfrak{g}$-module. So we assume that $m \equiv n \mod 2$. As before, the whole point is to find v'_n satisfying the condition 12.3(1), for $c = n$.

As above, there is a constant a such that

$$h\,.\,v'_n = n\,.\,v'_n + a\,.\,v_n\,. \tag{1}$$

We want to prove v'_n may be chosen so that $a = 0$. As in 12.4, we see that

$$h\,.\,v'_{n-2i} = (n-2i)\,.\,v'_{n-2i} + a\,.\,v_{n-2i} \qquad (i \geq 0)\,. \tag{2}$$

The weights in V_n are contained in $[n, -n]$, so the projection of $f\,.\,v'_{-n}$ in V_n is zero and we have, for some constant c,

$$f\,.\,v'_{-n} = c\,.\,v_{-n-2}\,. \tag{3}$$

Let $v''_n = v'_n - c\,.\,v_n$ and following 12.2(4), define v''_{n-2i} inductively by the relation

$$v''_{n-2i} = -i\,.f\,.\,v''_{n-2i+2} \qquad (i = 1, \dots, n)\,. \tag{4}$$

By induction on i, we see that

$$h\,.\,v''_{n-2i} = (n-2i)\,.\,v''_{n-2i} + a\,.\,v_{n-2i} \qquad (i = 0, \dots, n) \tag{5}$$

and also, in view of (3), that

$$f\,.\,v''_{-n} = f\,.\,v'_{-n} - c\,.f\,.\,v_{-n} = 0\,. \tag{6}$$

For $i = n$, the equality (5) gives

$$h\,.\,v''_{-n} = -n\,.\,v''_{-n} + a\,.\,v_{-n}\,. \tag{7}$$

Apply now f to both sides and recall that $f.h = h.f + 2f$. In view of (6) and 12.2(3) for $m = n$, we get

(8) $$a.(n+1).v_{-n-2} = 0.$$

But $n < m$ so $v_{-n-2} \neq 0$, whence $a = 0$.

We may therefore assume that v'_n is an eigenvector of h. There is no eigenspace for h with eigenvalue $n+2$ in V'_n, hence

(9) $$e.v'_n = b.v_{n+2} \qquad (b \in \mathbf{C}).$$

By 12.2(4), for $i = (m-n)/2$ (recall that $m \equiv n$ (2)), we get

(10) $$e.v_n = \big((m+n+2)/2\big).v_{n+2},$$

therefore

(11) $$w_n = v'_n - b\big((m+n+2)/2\big)^{-1}.v_n$$

satisfies the conditions

(12) $$h.w_n = n.w_n, \quad e.w_n = 0,$$

so that, by 12.3, $\mathfrak{g}.w_n$ is a copy of V_n complementary to V_m.

13. Fano's proof:

It deals with projective transformations and uses algebraic geometry. Given a finite dimensional vector space F over $\mathbf{C}$, we let $\mathbf{P}(F)$ be the projective space of one-dimensional subspace of F. If F is of dimension n, $\mathbf{P}(F)$ is isomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$.

13.1. The proof is contained in §§7, 8, 9 of [F]. §9 shows how to reduce it to the case considered in §3, a), b), that is, to the case of a short exact sequence 12.4(1) with $m \geq n$, but expressed in projective language, namely:

The space $\mathbf{P} = \mathbf{P}(E)$ contains a minimal irreducible invariant projective subspace $W = \mathbf{P}(V_m)$ of dimension m and the induced projective representation in the space W' of projective $(m+1)$-subspaces containing W is irreducible.

The problem is then to find an invariant projective subspace D not meeting W. If so, it has necessarily dimension n and $\mathbf{P}(E)$ is the join of W and D. Moreover, by the remark b) in §3, it may be assumed that $m \geq n$. Let us write N for the dimension of $\mathbf{P}$. Then $N = m+n+1$ and $m \geq (N-1)/2$.

As in 12.2(b), U is the one-parameter subgroup of $G = SL_2(\mathbf{C})$ generated by e. Its fixed point set is also the subspace E^e of E annihilated by e. Since

U is unipotent, any line invariant under U is pointwise fixed, so that the projective subspace $\mathbf{P}(E^U)$ associated to E^U is also the fixed point set $\mathbf{P}(E)^U$ of U on $\mathbf{P}(E)$. Similarly, it may be identified with the set $\mathbf{P}(E)^e$ of zeros of the vector field on $\mathbf{P}(E)$ defined by the action of U.

In §7, Fano proves that $\mathbf{P}(E)^e$ is a projective line. I am not sure I understand his argument, so I shall revert to the linear setup. As just pointed out, we have to show that E^e is two-dimensional.

In V_m and V_n it is one-dimensional, so the exact sequence 12.4(1) shows that $\dim E^e \leq 2$. As in §3, let E^* be the contragredient representation to E. Then E^{*^e} is the dual space to E/eE so it is equivalent to prove that $\dim E^{*^e} = 2$. Therefore we may assume that $m \leq n$ (our assumption earlier, but not the one of Fano). Fix a vector $v' \in E$ projecting onto a highest weight vector in V_n. It is an eigenvector of h if $m < n$, is annihilated by $(h - n\,.\,I)^2$ otherwise, and in both cases is annihilated by e (see 12.1 (4), (5)).

13.2. The next and main part of Fano's argument depends on some properties of the "rational normal scrolls", which we now recall (see [GH], p. 522–527). Assume $N \geq 2$ and let Z be a surface in $\mathbf{P}$, not contained in any projective subspace. Then its degree is at least $N-1$ ([GH], p. 173). Those of degree $N-1$ have been classified, up to projective transformations ([GH], *loc.cit.*). Only one is not ruled, the Veronese embedding of $\mathbf{P}_2(\mathbf{C})$ in $\mathbf{P}_5(\mathbf{C})$.

The others are the *rational normal scrolls* $S_{a,b}$ $(a+b = N-1)$, obtained in the following way: Fix two independent projective subspaces A, B of dimension a, b. Then $\mathbf{P} = A * B$ is the join of A and B. Let C_A (resp. C_B) be a rational normal curve in A (resp. B) and $\varphi : C_A \to C_B$ an isomorphism. Then $S_{a,b}$ is the space of the lines $D(x, \varphi(x))\,(x \in C_A)$. If $a > 0$, but $b = 0$, then C_B is a point, φ maps C_A onto a point and $S_{a,b}$ is the cone over C_A with vertex C_B. It has a unique singular point, namely C_B and this is the only case where $S_{a,b}$ is not smooth ([GH], p. 525).

A rational curve in $S_{a,b}$ which cuts every line $D(x, \varphi(x))$ in exactly one point is called a directrix. By construction C_A (resp. C_B) is a directrix of degree a (resp. b). The main result used by Fano is that if $a > b$, then C_B is the unique directrix of degree b ([GH], p. 525). Fano deduces this essentially from an earlier result of C. Segre [Se].

If $a = b$, then we may identify A to B by a map φ which takes C_A to C_B. It is clear that in that case $S_{a,b} = S_{a,a} = \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.

13.3. We now come back to the situation in 13.1. In W there is exactly one rational normal curve C stable under G. The zero set $\mathbf{P}(E)^e$ of e is a line (13.1) and $\mathbf{P}(E)^e \cap C$ consists of one point, namely, W^e. Let Z be the set of transforms $g \cdot \mathbf{P}(E)^e$ of $\mathbf{P}(E)^e$, $(g \in G)$. Since $\mathbf{P}(E)^e$ is stable under the upper triangular group B and G/B is complete (in fact a smooth rational curve), Z is a projective subvariety, a G-stable ruled surface. We first dispose of a special case. The line $g.P(E)^e$ is the fixed point set of the subgroup ${}^gU = g.U.g^{-1}$, conjugate to U by g. Assume that two distinct such lines have a common point. It would then be fixed by two distinct conjugates of U. But it is immediate that two such subgroups generate G, so that there would be a fixed point D of G in $\mathbf{P}(E)$, necessarily outside W. Then $\mathbf{P}(E)$ would be the join of W and D, and we would be through. From now on, we assume that the lines $g.\mathbf{P}(E)^e$ either coincide or are disjoint. We want to prove that Z has degree $N-1$ in $\mathbf{P}(E)$. First we claim that it is not contained in any hyperplane Y of $\mathbf{P}(E)$. Indeed, if it were, it would be contained in a G-stable proper subspace F, the intersection of the transforms of Y. The subspace F would contain W properly, which would contradict the irreducibility of the quotient representation in $\mathbf{P}(V_n)$. The degree of Z is therefore at least $N-1$ ([GH], p. 173–4). There remains to show that it is $\leq (N-1)$.

Let $C' \subset W'$ be the closed orbit of G, which plays the same role as C in W. In particular, it has degree n. Let now Y be a generic hyperplane of $\mathbf{P}(E)$ among those containing W. Viewed as a hyperplane in W', it cuts C' in n distinct points Q_i $(i = 1, \dots, n)$. Let U_i be the conjugate of U which fixes Q_i (see 12.2, (b)). The intersection $Z \cap Y$ is a (reducible) curve. We want to prove it has degree $N-1$ in Y. We claim first

$$(1) \qquad Y \cap Z = C \cup D_1 \cup \cdots \cup D_n \qquad (D_i = \mathbf{P}(E)^{U_i}),$$

where the D_i are disjoint projective lines, each intersecting C at exactly one point.

First, by construction, $C \subset Z \cap Y$, in fact $C = W \cap Z \cap Y$. Let $x \in Z \cap Y$, $x \notin W$. It belongs to some line $D_g = g.\mathbf{P}(E)^e$. The line D_g also contains $g.W^e$, which belongs to $Z \cap Y$, too. Therefore $D_g \subset Y$, and of course $D_g \subset Z$, hence $Z \cap Y$ is the union of C and some of the lines D_g. The line D_g spans with W a projective subspace of dimension equal to $\dim W + 1$, which represents a point of W', fixed under ${}^gU$. It belongs therefore to Y if and only ${}^gU$ is one of the U_i, i.e. if and only if D_g is one of the D_i's and (1) follows.

Since C has degree $m = \dim W$ in W, it follows that $Z \cap Y$ is a curve of degree $m+n$ in Y, hence Z is a surface of degree at most $m+n = N-1$ in $\mathbf{P}(E)$.

Thus Z is a ruled surface, not contained in a hyperplane, of smallest possible degree. It is therefore a "rational normal scroll" (13.2). It is isomorphic to $S_{a,b}$ where $a = \dim W = m$ and $b = N - 1 - a = n$.

Recall that we have reduced ourselves to the case $a \geq b > 0$. Assume first $a > b$. Then, (see 13.2), Z contains a *unique* directrix of degree b. It is a normal curve in a b-dimensional subspace, which must be invariant under G, since Z is. This provides the complementary subspace to W.

Let now $m = n$. Then (13.2), $Z = C \times C'$ is a product of two copies of $\mathbf{P}^1(\mathbf{C})$, where C is, as before, a G-stable rational normal curve in W and $C' = \mathbf{P}(E)^e$. The transforms $g\,.\,\mathbf{P}(E)^e$ of C' are the lines $\{c\} \times C'\,(c \in C)$.

The lines $C_y = C \times \{y\}$ $(y \in C')$ are "directrices". We claim that they are all invariant under G. Clearly, the intersection number $C_y \cdot C_z$ is zero if $y \neq z$ $(y, z \in C')$. Let $g \in G$. Since it is connected to the identity, we have then also $(g\,.\,C_y) \cdot C_z = 0$, therefore $g\,.\,C_y \cap C_z = \varnothing$ unless $g\,.\,C_y = C_z$. Since $g\,.\,C_y$ must meet at least one C_z, we have $g\,.\,C_y = C_z$ for some z and we see that G permutes the curves $\{C_y\}\,(y \in C')$. Each such curve contains a fixed point of e, hence of U. Therefore C_y is stable under U. Now the subgroup H of G leaving each curve C_y stable is a normal subgroup, which is $\neq \{1\}$ since it contains U. But G is a simple Lie group, therefore $H = G$, which proves our contention. Any curve C_y is a rational normal curve in a subspace W'_y which is necessarily G-stable. This provides infinitely many G-invariant subspaces and concludes the proof.

REMARK. Let us compare the orders of the steps in the proofs of Cartan and of Fano. In 12.4 and 12.5 the first item of business is to show that the action of h on a certain h-stable two-dimensional subspace is diagonalisable. That space is E^e in 12.4, and subsequently shown to be E^e in 12.5. Once a new eigenvector of h annihilated by e is found, 12.3 can be used. In Fano, the first step is to show that E^e is two-dimensional or rather, equivalently, that $\mathbf{P}(E)^e$ is a projective line. There, the analogue of the first step of Cartan would be to prove the existence of *two* fixed points on $\mathbf{P}(E)^e$ of h, or of the group $H = \{e^{th}\}$ generated by h. One is W^e. In the generic case $m > n$, Fano's argument may also be viewed as a search for this second fixed point: it is the intersection of $\mathbf{P}(E)^e$ with the (unique) directrix C_B. However, since the proof provides directly the G-orbit C_B of that second fixed point, the argument is not phrased in that way.

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