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Artikel: POLYNOMIALS MODULO p WHOSE VALUES ARE SQUARES (ELEMENTARY IMPROVEMENTS ON SOME CONSEQUENCES OF WEIL'S BOUNDS)

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Kapitel: §2. Main arguments

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Define $m(p)$ as the minimal positive integer m such that $p^m > m2^p$. We have $m(p) \sim p\log 2/\log p$. In §3.3, we shall show in a simple way that $d(p) \leq 2m(p)$ (perhaps an essentially optimal bound). Proving good lower bounds for $d(p)$ is more difficult. With the help of (1) it is easy to show that $d(p) > \sqrt{p}$. This is essentially the best that we can extract from (1). In fact, we have already remarked that (1) does not provide any information for $d > 3 + \sqrt{p}$. Here we give a short elementary proof of the following

THEOREM. *We have* $d^2(p) + 3d(p) \geq 2p + 2$, *hence* $d(p) \geq \sqrt{2p} - \frac{3}{2}$.

An immediate corollary is that the number of solutions in $\mathbf{F}_p^2$ of $y^2 = f(x)$ with $y \neq 0$, is at least $\sqrt{2p} - \frac{3}{2} - d$, provided $f \in \mathbf{F}_p[X]$ has degree d and at least one simple root. In fact, let

$$S := \{u \in \mathbf{F}_p \ : \ f(u) \text{ is a nonzero square in } \mathbf{F}_p\}$$

and put $g(X) := \prod_{u \in S}(X-u)$. Then observe that if a is a quadratic non-residue mod p, the polynomial $g(X)^2 af(X)$ assumes only square values on $\mathbf{F}_p$, without being a square. The theorem implies $2\deg g + d \geq \sqrt{2p} - \frac{3}{2}$. On the other hand, $2\deg g$ is precisely the number of solutions we are considering. We shall outline in §3.2 how to improve on this bound.

§2. MAIN ARGUMENTS

We start with a simple example to outline the origin of the method. We give a self-contained nine-line proof of the following claim: *Let* $q = 2r + 1 > 3$ *be an odd prime power and let* $f \in \mathbf{F}_q[X]$ *be a cubic polynomial. Then the equation* $y^2 = f(x)$ *has at least one solution* $(x_0, y_0) \in \mathbf{F}_q^2$.

(Mordell [Mo, p. 41] had to invoke fairly complicated arguments even to deal with the special case $f(X) = X^3 + k$.)

Assume the assertion false. Then $f(u)^r = -1$ for all $u \in \mathbf{F}_q$. Hence every element of $\mathbf{F}_q$ is a root of $f(X)^r + 1$ and so, identically,

$$(2) \qquad f(X)^r + 1 = (X^q - X)S(X),$$

where $S \in \mathbf{F}_q[X]$ has degree $3r - q = r - 1$. Differentiating the equation we get

$$(3) \qquad rf'(X)f(X)^{r-1} = (X^q - X)S'(X) - S(X).$$

Multiply (2) by $rf'(X)$, (3) by $f(X)$ and subtract to obtain

$$(4)\qquad rf'(X) = (X^q - X)(rf'(X)S(X) - f(X)S'(X)) + f(X)S(X).$$

Observe now that $rf'(X)-f(X)S(X)$ has degree $3+\deg S = r+2$ and is divisible by $X^q - X$, in view of (4). Hence $r+2 \geq q = 2r+1$, i.e. $r \leq 1$ and $q \leq 3$. □

We now prove the theorem. Suppose that $f \in \mathbf{F}_p[X]$ $(p > 3)$ has degree $d \leq p-3$, is not a square in $\mathbf{F}_p[X]$ but assumes on $\mathbf{F}_p$ only values which are squares in $\mathbf{F}_p$. Write $f(X) = a\prod_{i=1}^{h} f_i(X)^{m_i}$, where $a \in \mathbf{F}_p^*$, the $f_i \in \mathbf{F}_p[X]$ are distinct monic irreducible polynomials and the m_i are positive integers. Factoring out suitable even powers of the f_i, we may assume[2]) that $1 \leq m_i \leq 2$. Since $d < p$, there exists $u \in \mathbf{F}_p$ with $f(u) \neq 0$, so $f(u)$ is a nonzero square in $\mathbf{F}_p$. If all the m_i were even, then a would be a nonzero square in $\mathbf{F}_p$ and f would be a square in $\mathbf{F}_p[X]$, contrary to assumptions. Therefore at least one of the m_i is equal to 1, proving that f has at least a simple root α (in some finite field).

Let now $u \in \mathbf{F}_p$. Then, writing $p = 2r+1$, either $f(u) = 0$ or $f(u)^r = 1$. Therefore $f(X)(f(X)^r - 1)$ is divisible by $X^p - X$. We write

$$(5)\qquad f(X)^{r+1} - f(X) = (X^p - X)S(X),$$

where $S \in \mathbf{F}_p[X]$ has degree $(r+1)d - p$. Differentiate (5) to obtain

$$(6)\qquad (r+1)f'(X)f^r(X) - f'(X) = (X^p - X)S'(X) - S(X).$$

Similarly to the above example, multiply (5) by $(r+1)f'(X)$, (6) by $f(X)$ and subtract. The result is

$$(7)\qquad f(X)S(X) = (X^p - X)(f(X)S'(X) - (r+1)f'(X)S(X)) - rf(X)f'(X).$$

This equation is the first step in a recursion that we are going to construct. Define the differential operators Δ_m on $\mathbf{F}_p[X]$ by setting, for $\phi \in \mathbf{F}_p[X]$,

$$\Delta_m(\phi)(X) := f(X)\phi'(X) - (r+m+1)f'(X)\phi(X),$$

and put, for $m \geq 0$,

$$(8)\qquad \begin{cases} S_0(X) := S(X), & S_{m+1}(X) := \Delta_m(S_m)(X), \\ R_0(X) := -rf(X)f'(X), & R_{m+1}(X) := \Delta_{m+1}(R_m)(X). \end{cases}$$

Then (7) reads

$$(9)\qquad f(X)S_0(X) = (X^p - X)S_1(X) + R_0(X).$$

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[2]) Note that when m_i is even we cannot factor out $f_i(X)^{m_i}$ without danger of destroying the properties of $f(X)$. In fact we could have a priori $f(u) = f_i(u) = 0$ for some $u \in \mathbf{F}_p$ while $(f/f_i^{m_i})(u)$ could be a non-square in $\mathbf{F}_p$. It is however safe to factor out $f_i^{m_i-2}$.

We shall prove by induction that for all $m \geq 0$ we have

$$(m+1)f(X)S_m(X) = (X^p - X)S_{m+1}(X) + R_m(X). \tag{10}$$

For $m = 0$ this is just (9). Assume (10) true and apply to both sides the operator Δ_m. Note that $\Delta_m(\phi\psi) = \phi\Delta_m(\psi) + \phi' f \psi$. We obtain

$$(m+1)f\Delta_m(S_m) + (m+1)f'fS_m = (X^p - X)\Delta_m(S_{m+1}) - fS_{m+1} + \Delta_m(R_m).$$

Now use (10) to substitute for $(m+1)fS_m$ in the second term of the left side. We get

$$(m+1)fS_{m+1} + f'((X^p - X)S_{m+1} + R_m) = (X^p - X)\Delta_m(S_{m+1}) - fS_{m+1} + \Delta_m(R_m),$$

whence

$$(m+2)fS_{m+1} = (X^p - X)\big(\Delta_m(S_{m+1}) - f'S_{m+1}\big) + \Delta_m(R_m) - f'R_m.$$

Now, to conclude the inductive argument we have only to note that $\Delta_m(\phi) - f'\phi$ equals just $\Delta_{m+1}(\phi)$.

Recall that f has a simple root α. We continue by proving the following

CLAIM. *Let $m \leq r$. Then α cannot be a double root of S_m. In particular, $S_m(X) \neq 0$ for $m \leq r$.*

For $m = 0$ this follows at once from (5). Suppose the claim true for a certain m and assume by contradiction that α is a double root of $S_{m+1}(X) = f(X)S_m{}'(X) - (r+m+1)f'(X)S_m(X)$, where $m+1 \leq r$. Then, first of all we would have $(r+m+1)f'(\alpha)S_m(\alpha) = 0$. This implies that $S_m(\alpha) = 0$, since $f'(\alpha) \neq 0$ and since $r+m+1 \leq 2r = p-1$. Next, we compute

$$S_{m+1}{}'(X) = f'(X)S_m{}'(X) + f(X)S_m{}''(X) \\ - (r+m+1)f''(X)S_m(X) - (r+m+1)f'(X)S_m{}'(X).$$

Since $f(\alpha) = S_m(\alpha) = S_{m+1}{}'(\alpha) = 0$, we obtain that $-(r+m)f'(\alpha)S_m{}'(\alpha) = 0$. As before, this implies that $S_m{}'(\alpha) = 0$. Hence α would be a double root of $S_m(X)$, a contradiction to the inductive assumption.

As in the example, we shall conclude by comparison of degrees. Define

$$\rho_m := \deg R_m, \qquad \sigma_m := \deg S_m,$$

where we may agree that the zero polynomial has degree $-\infty$. We have $\rho_0 = 2d - 1$ and we derive directly from the recursion formulae (8) that $\rho_{m+1} \leq \rho_m + d - 1$, whence

$$\rho_m \leq d + (m+1)(d-1)\,. \tag{11}$$

Also, from (5), (10) and (11) we get (recalling our definition of $\deg 0$),

$$\begin{cases} \sigma_0 = (r+1)d - p \\ \sigma_{m+1} \leq \max(\sigma_m + d, \rho_m) - p \leq \max(\sigma_m, (m+1)(d-1)) + d - p\,. \end{cases} \tag{12}$$

Observe that we have $\sigma_0 = (r+1)d - p = (r+1)d - (2r+1) = (d-2)r + (d-1) \geq d-1$. Suppose that the inequality

$$\sigma_m \geq (m+1)(d-1) \tag{13}$$

is true for $m = 0, \ldots, M-1$, but not for $m = M$ (possibly $M = \infty$). Then $M \geq 1$. Moreover, by (12) we have $\sigma_{m+1} \leq \sigma_m + d - p$ for $m \leq M-1$, whence

$$\sigma_m \leq \sigma_0 + m(d-p) = rd - (m+1)(p-d), \qquad \text{for } m \leq M\,. \tag{14}$$

Applying (13) and (14) with any $m \leq M-1$, we get $rd - (m+1)(p-d) \geq (m+1)(d-1)$, i.e. $2r(m+1) \leq rd$. Therefore we have

$$M \leq \frac{d}{2}\,. \tag{15}$$

Finally, apply (12) for $m = M$ and observe that $M \leq d/2 \leq r-1$, hence $S_{M+1} \neq 0$ by the Claim. We obtain $0 \leq \sigma_{M+1} \leq (M+1)(d-1) + d - p$, whence, comparing with (15),

$$2p \leq \begin{cases} d^2 + 3d - 2 & \text{if } d \text{ is even} \\ d^2 + 2d - 1 & \text{if } d \text{ is odd}\,. \end{cases}$$

This proves the theorem and more. □

§3. Remarks

(1) The method gives some information also in the case of a general finite field $\mathbf{F}_q$. The same arguments as above work everywhere, on replacing p by q, except that in the Claim we must now suppose that $m \leq r_0$, where $p = 2r_0 + 1$. The final conclusion will be that $d \geq \min(r_0, \sqrt{2q} - (3/2))$. This is still sufficient to prove that equations $y^2 = f(x)$ in $\mathbf{F}_q$ have some solution, provided p is sufficiently large compared to $\deg f$.

(2) The same method of proof produces a lower bound for the number N' of solutions of $y^2 = f(x)$ such that $y \neq 0$. This bound is better than the one which has been stated above, as a corollary of the theorem itself. To