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THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS

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ABSTRACT. The aim of this paper is to investigate the connection between some generalizations of the three distance theorem and combinatorics on words for sequences defined as codings of irrational rotations on the unit circle. We also give some new results concerning the frequencies of factors for such sequences.

1. INTRODUCTION

For a given α in $]0,1[$, let us place the points $\{0\}, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ on the *unit circle* (we mean here the circle of *perimeter* 1), where $\{x\}$ denotes as usual, the fractional part of x (i.e., if $\lfloor x \rfloor$ denotes the largest integer not exceeding x, $\{x\} = x - \lfloor x \rfloor$). These points partition the unit circle into $n+1$ intervals having at most three lengths, one being the sum of the other two. This property is known as the *three distance theorem* and can be seen as a geometric interpretation of good approximation properties of the Farey partial convergents in the continued fraction expansion of α.

The connection between this classical theorem in diophantine approximation and combinatorics on words is particularly apparent in the following result, known as the *three gap theorem*, which is equivalent to the three distance theorem and can be seen as its "dual": assume we are given α and β in the interval $]0,1[$, the gaps between the successive n for which $\{\alpha n\} < \beta$ take at most three values, one being the sum of the other two. It is indeed natural to introduce the binary sequence with values 0 and 1, defined as the coding of the orbit of a point of the unit circle under the rotation by angle α with respect to the intervals $[0,\beta[$, $[\beta,1[$ (in particular, if β equals $1-\alpha$ or α, this sequence is a Sturmian sequence): the lengths of strings consisting of 0's and 1's are thus directly connected with the three gaps. In fact, the three distance and the three gap theorems have deep relations with the measure-theoretic and topological properties of the dynamical systems associated with codings of rotations.

The aim of this paper is to review the different generalizations of the three distance and three gap theorems and to emphasize the relationships with combinatorics on words. This paper is organized as follows. We recall in Section 2 basic definitions and properties concerning codings of rotations. We emphasize the connections between frequencies of factors of given length for such sequences and the lengths of the intervals obtained by partitioning the unit circle by a set of points in arithmetical progression. We prove in particular that the three distance theorem is equivalent to the fact that the frequencies of factors of given length of a Sturmian sequence take at most three values. Furthermore, this last statement is easily proved by using the notion of graph of words, which gives us a very simple combinatorial proof of the three distance theorem. Section 3 is devoted to the study of the three distance theorem. We introduce the three gap theorem in Section 4. We will deduce from these two theorems, in Section 5, the expression of the recurrence function of a Sturmian sequence, due to Hedlund and Morse [40]. Section 6 deals with generalizations of the three distance and the three gap theorems. We give in Section 7 a direct proof of a particular case of the two-dimensional version of the three distance theorem (i.e., that there are at most 5 lengths when the unit circle is partitioned by the points $\{i\alpha\}$ and $\{i\alpha+\beta\}$, for $0 \le i \le n$). In Section 8, we give a proof of the $3d$ distance theorem, proved by Chung and Graham [18, 37] (i.e., that there are at most $3d$ lengths when the unit circle is partitioned by the points $\{k_i\alpha+\gamma_i\}$, for $0 \le i \le d$ and $0 \le k_i \le n_i$).

In each case, we study the connection with frequencies of codings of rotations. More precisely, we prove that the frequencies of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values and we deduce from the two-dimensional version of the three distance theorem that the frequencies of a coding of an irrational rotation with respect to a partition in d intervals of the same length take ultimately at most $d+3$ values; more generally, we prove that the frequencies of a coding of an irrational rotation with respect to a partition into d intervals (not necessarily of the same length) take ultimately at most $3d$ values (this result corresponds to the $3d$ distance theorem).

Let us first review some of the many related results and applications of the three distance theorem. We will focus on the theorem itself and its different proofs in Section 3.

As one of the first applications the theorem of Hartman [33] (which answers an earlier question of Steinhaus concerning the circular dispersion spectrum) has been proved in [53].

THEOREM 1. *Let $0 < \alpha < 1$ be an irrational number and let n be a positive integer. Let H_n (respectively, h_n) denote the maximal (respectively, the minimal) length of the $n+1$ intervals obtained by partitioning the unit circle by the points of the set $\{i\alpha,\ 0 \le i \le n-1\}$. If the partial quotients of the regular continued fraction expansion of α are unbounded, then*

$$\liminf_{n\to+\infty} n \cdot h_n = 0\,,$$

$$\limsup_{n\to+\infty} n \cdot h_n = 1\,,$$

$$\liminf_{n\to+\infty} n \cdot H_n = 1\,,$$

$$\limsup_{n\to+\infty} n \cdot H_n = +\infty\,.$$

In [21] Deléglise studies the length $L(h)$ of the smallest closed interval I of the unit circle such that $I, 2I, \ldots, hI$ cover the circle. More precisely, he shows the following result.

THEOREM 2. *Let I be a closed interval of minimal length L such that $I, 2I, \ldots, hI$ cover the circle; we have, for $h \ge 3$,*

$$L = \begin{cases} 3/(h(h+2)), & \text{if } h \equiv 0 \text{ or } 1 \mod 3\,, \\ 3/(h(h+2)-2), & \text{if } h \equiv 2 \mod 3\,. \end{cases}$$

In particular, the function $L(h)$ is equivalent to $3/h^2$ when h tends towards infinity.

In [7], Bessi and Nicolas apply the three distance theorem to 2-highly composite numbers, i.e., if $\mathcal{N}_2$ denotes the set of integers having only 2 and 3 as prime factors, an integer n in $\mathcal{N}_2$ is said to be a 2-highly composite number if for any m in $\mathcal{N}_2$ such that $m < n$, then the number of divisors of m is strictly less than n. They prove, in particular, that there exists a constant c such that the number of 2-highly composite numbers smaller than X is larger than $c(\log X)^{4/3}$.

In [8] Boshernitzan extends the three distance theorem to the case of interval exchange maps in his proof of Keane's conjecture, which states the unique ergodicity of Lebesgue almost all minimal interval exchange maps. Let us recall briefly the definition of an interval exchange map. Assume we are given $\lambda = (\lambda_1, \ldots, \lambda_r)$ in the positive cone in $\mathbf{R}^r$, i.e., $\lambda_i > 0$, for $1 \le i \le r$; it defines a segment $I = [0, \sum_1^r \lambda_k[$ of $\mathbf{R}$ composed of r intervals $I_i = [\sum_0^i \lambda_k, \sum_1^{i+1} \lambda_k[$, for $0 \le i \le r-1$ and by taking $\lambda_0 = 0$. Let σ denote a permutation of $\{1, 2, \ldots, r\}$. The interval exchange map T associated with λ and σ is defined as the map from I to I which exchanges the intervals I_i according to the permutation σ:

$$T(x) = x + \Big(\sum_{j<\sigma(i)} \lambda_{\sigma^{-1}(j)} - \sum_{j<i} \lambda_i \Big), \quad \text{for } x \in I_i .$$

The n-fold iterate of T is also an interval exchange map of say $r(n)$ intervals $I_1, \ldots, I_{r(n)}$. Boshernitzan proved the following

THEOREM 3. *The number of intervals* $I_1, \ldots, I_{r(n)}$ *of different length is not greater than* $3(r-1)$, *for all* $n \ge 1$.

Let us note that a two interval exchange map is a rotation; hence when $r = 2$, this theorem reduces to the three distance theorem.

As another ergodic application, we have the following. In [5] (see also [12]) topological and measure-theoretic covering numbers (i.e., the maximal measure of Rokhlin stacks having some prescribed regularity properties) are computed first for the symbolic dynamical systems associated to the rotation of argument α acting on the partition of the circle by the point β and then to exchange of three intervals; in this way, it is proved that every ergodic exchange of three intervals has simple spectrum, and a new class of exchanges of three intervals having nondiscrete spectrum is built. Results for irrational rotations of the torus $\mathbf{T}^2$ can also be obtained, by replacing intervals by Voronoï cells (see [16]).

The connections between Beatty sequences and the three distance and three gap theorems, and more precisely with the gaps in the intersection of Beatty sequences, have been investigated by Fraenkel and Holzman in [26]. We will discuss their results in Section 6.

J. Shallit introduces in [47] a *measure of automaticity* of a sequence. This measure counts the number of states in a minimal deterministic finite automaton which generates the prefix of size n of this sequence. Let us recall that a sequence has a finite measure of automaticity if and only if this sequence is a letter-to-letter projection of a fixed point of a constant length morphism of a free monoid. The author deduces in [47] a measure of automaticity of Sturmian codings of rotations from the three distance theorem, which are shown to have a high automaticity measure, even when they are fixed points of homomorphism.

THEOREM 4. *Let* $0 < \alpha < 1$ *be an irrational number with bounded partial quotients. Let* $u_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$, *for* $n \geq 1$. *The automaticity of the sequence* $(u_n)_{n \geq 1}$ *has the same order of magnitude as* $n^{1/5}$.

Let us also note the following two applications of the three distance theorem in theoretical computer science. The first one deals with multiplicative hashing, as Fibonacci hashing, and is quoted in [34]. The second one is due to Lefèvre and gives a fast algorithm for computing a lower bound on the distance between a straight line and the points of a regular grid. This algorithm is used to find worst cases when trying to round the elementary functions correctly in floating-point arithmetic (see [36]).

Langevin studies in [35] the three distance theorem in connection with a mathematical model of the ventricular parasystole. He proves in particular the following generalization of the three distance theorem to lattices.

THEOREM 5. *Let* L *be a lattice in* $\mathbf{R}^2$. *Let* I *be a bounded interval of* $\mathbf{R}$ *and let* $L(I) = \{(x, y),\ x \in I\} \cap L$. *For any point* M *of* $L(I)$, *let* $S(M)$ *denote the smallest point* $M' \neq M$ *of* $L(I)$ *such that* M *is smaller than* M', *in lexicographic order. Then there exists a basis* (U, V) *of the lattice* L *such that for any point* M *of* $L(I)$, *the difference* $S(M) - M$ *is either equal to* U, V *or* $U + V$.

Let us note that this theorem has been generalized by Fried and V. T. Sós to groups in [28].

Finally, van Ravenstein studies in [43] the phenomenon of phyllotaxis, i.e., the regular leaf arrangement, which is given by the Fibonacci phyllotaxis for most plants (see also the work of Marzec and Kappraff in [38]). In [42] van Ravenstein also applies the three distance theorem to evaluate some values of the discrepancy of the sequence $(n\alpha)_{n \in \mathbf{N}}$, for α irrational.