

Zeitschrift: L'Enseignement Mathématique
Band: 44 (1998)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THREE DISTANCE THEOREMS AND COMBINATORICS ON WORDS
Kapitel: 4.2 Applications
Autor: Alessandri, Pascal / Berthé, Valérie
DOI: <https://doi.org/10.5169/seals-63900>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 20.11.2024

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

THEOREM 9. *Let $u = (u_n)_{n \in \mathbb{N}}$ be a coding of the rotation by irrational angle α . Suppose that there exists an interval of \mathcal{P} of length $L > \sup(\alpha, 1 - \alpha)$. Let $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$ and $(c_k)_{k \in \mathbb{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi,$$

with $k \geq 1$, $0 < \psi \leq \eta_k$ and $1 \leq m \leq c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$\begin{aligned} n^{(1)} &= q_{k+1} - (m - 1)q_k - 1, \text{ if } \psi \neq \eta_k, \\ n^{(1)} &= q_{k+1} - mq_k - 1, \text{ if } \psi = \eta_k \text{ and } m < c_{k+1}, \\ n^{(1)} &= q_k - 1, \text{ if } \psi = \eta_k \text{ and } m = c_{k+1}. \end{aligned}$$

4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. *Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[\}$. The frequencies of factors of u of length $n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1 - \alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n - 1, \quad 0 \leq i \leq p - 1.$$

The complexity of a coding on p letters of an irrational rotation ultimately has the form $p(n) = an + b$, where $a \leq p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. *Let $u = (u_n)_{n \in \mathbb{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[\}.$$

Let $(k_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$k_0 = p = \text{card}(F),$$

$$k_n = \text{card} \{ \beta_i \in F; \forall k \in [1, \dots, n], R^{-k}(\beta_i) \notin F \}.$$

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let $n^{(1)}$ denote the connectedness index of u .

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b.$$

REMARKS.

- Note that if $1, \alpha, \beta_1, \dots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.

- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n \in \mathbf{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n, n \in \mathbf{N}\} = \{u, \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\}.$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ ; the characteristic sequence $(v_n)_{n \in \mathbf{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$