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THEOREM 9. *Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of the rotation by irrational angle α. Suppose that there exists an interval of $\mathcal{P}$ of length $L > \sup(\alpha, 1-\alpha)$. Let $\left(\frac{p_k}{q_k}\right)_{k\in\mathbf{N}}$ and $(c_k)_{k\in\mathbf{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi ,$$

with $k \geq 1$, $0 < \psi \leq \eta_k$ and $1 \leq m \leq c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$n^{(1)} = q_{k+1} - (m-1)q_k - 1, \text{ if } \psi \neq \eta_k ,$$
$$n^{(1)} = q_{k+1} - mq_k - 1, \text{ if } \psi = \eta_k \text{ and } m < c_{k+1} ,$$
$$n^{(1)} = q_k - 1, \text{ if } \psi = \eta_k \text{ and } m = c_{k+1} .$$

4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. *Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. The frequencies of factors of u of length $n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1-\alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n-1, \ 0 \leq i \leq p-1 .$$

The complexity of a coding on p letters of an irrational rotation ultimately has the form $p(n) = an + b$, where $a \leq p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. *Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\} .$$

Let $(k_n)_{n\in\mathbf{N}}$ be the sequence defined by

$$k_0 = p = \mathrm{card}(F) ,$$

$$k_n = \mathrm{card}\left\{\beta_i \in F;\ \forall k \in [1, \ldots, n],\ R^{-k}(\beta_i) \notin F\right\} .$$

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a) .$$

Let $n^{(1)}$ denote the connectedness index of u.

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b .$$

REMARKS.

• Note that if $1, \alpha, \beta_1, \ldots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.

• Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n \in \mathbf{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n \in \mathbf{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n,\ n \in \mathbf{N}\} = \{u,\ \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\} .$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ; the characteristic sequence $(v_n)_{n \in \mathbf{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$