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frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the $3d$ *distance theorem.* This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE $3d$ DISTANCE THEOREM. *Assume we are given* $0 < \alpha < 1$ *irrational,* $\gamma_1, \ldots, \gamma_d$ *real numbers and* $n_1, \ldots, n_d$ *positive integers. The points* $\{n\alpha + \gamma_i\}$, *for* $0 \leq n < n_i$ *and* $1 \leq i \leq d$, *partition the unit circle into at most* $n_1 + \cdots + n_d$ *intervals, having at most* $3d$ *different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \ldots, \eta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus and where $1, \eta_1 \ldots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to $\mathbf{T}^d$, where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points