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Autor: Alessandri, Pascal / Berthé, Valérie
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7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation R would have been with respect to the partition $[0, \beta[, [\beta, 1[$. The points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$ are the endpoints of the sets $I(w_1, \dots, w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1, \dots, w_n)$. Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

THEOREM 19. *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0, \beta[, [\beta, 1[$, where $0 < \beta < 1$. Let $n^{(1)}$ denote the connectedness index of u . The frequencies of factors of given length $n \geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1 \cdots a_n$ is a factor of the sequence u , then $a_n \cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof. It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n . Let s_n be the reflection of the circle defined by $s_n: x \rightarrow \{\beta - (n - 1)\alpha - x\}$. We have $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$, for $j = 0, 1$, following the previous notation. The image of $I(w_1, \dots, w_n)$ by s_n is $I(w_n, \dots, w_1)$; they thus have the same length, which gives the result.

REMARK. A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n + 1) - p(n) = 2$ can be found in [24] or in [46].

8. THE 3d DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the 3d distance theorem.

THE 3d DISTANCE THEOREM. *Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \dots, \gamma_d$ real numbers and n_1, \dots, n_d positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the unit circle into at most $n_1 + \cdots + n_d$ intervals, having at most 3d different lengths.*