8. The 3d distance theorem

7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation R would have been with respect to the partition $[0,\beta[, [\beta,1[$. The points $\{0\},\{\beta\},\{\alpha\},\{\beta+\alpha\},\ldots,\{n\alpha\}$, $\{\beta+n\alpha\}$ are the endpoints of the sets $I(w_1,\ldots,w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1,\ldots,w_n)$. Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

THEOREM 19. *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0,\beta[, [\beta,1[\}$, where $0<\beta<1$. Let $n^{(1)}$ denote the connectedness index of u. The frequencies of factors of given length $n\geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1\cdots a_n$ is a factor of the sequence u, then $a_n\cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof. It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n. Let s_n be the reflection of the circle defined by $s_n\colon x\to\{\beta-(n-1)\alpha-x\}$. We have $s_n(R^{-k}(I_j))=R^{(-n+1+k)}(I_j)$, for $j=0,1$, following the previous notation. The image of $I(w_1,\ldots,w_n)$ by s_n is $I(w_n,\ldots,w_1)$; they thus have the same length, which gives the result.

REMARK. A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n+1)-p(n)=2$ can be found in [24] or in [46].

8. THE $3d$ DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the $3d$ *distance theorem*.

THE $3d$ DISTANCE THEOREM. *Assume we are given $0<\alpha<1$ irrational, $\gamma_1,\ldots,\gamma_d$ real numbers and $n_1,\ldots,n_d$ positive integers. The points $\{n\alpha+\gamma_i\}$, for $0\leq n<n_i$ and $1\leq i\leq d$, partition the unit circle into at most $n_1+\cdots+n_d$ intervals, having at most $3d$ different lengths.*

Proof. Let us consider a coding of the rotation by angle α under the left-closed and right-open partition of the unit circle bounded by all the points of the form $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$; let $\beta_0, \ldots, \beta_{p-1}$ denote these consecutive points. The letter associated with the interval $I_k = [\beta_k, \beta_{k+1}[$ has a unique right extension, except when I_k contains points of the form $\{\beta_i - \alpha\}$. Suppose there are $q \geq 2$ points of this form; the associated letter has $q+1$ right extensions. Since there are at most d points of this type, we obtain $p(2) - p(1) \leq d$. We deduce from Theorem 6 that there are at most $3d$ different frequencies for the letters of the coding, i.e., there are at most $3d$ different lengths for the intervals I_k.

REMARK. The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point $\{\gamma_i\}$ (i.e., with one extension of a factor having more than one right extension) or with a finish point $\{(n_i - 1)\alpha + \gamma_i\}$ (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are $3d$ such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle $1 - \alpha$ under the partition $\{[\gamma_1, \gamma_2[, \ldots, [\gamma_d, \gamma_1[\}$. For such a coding, the $3d$ distance theorem can be rephrased as follows.

THEOREM 20. *The frequencies of the factors of given length $n \geq n^{(1)}$ of a coding of a rotation by irrational angle under a partition in d intervals take at most $3d$ values, where $n^{(1)}$ denotes the connectedness index.*

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