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## 3. EXPLICIT SOLUTIONS

In this section we find explicit solutions for the Lagrange top (2). We compute first the Baker-Akhiezer function of the  $\mathfrak{sl}(2, \mathbf{C})$  (or rather  $\mathfrak{su}(2)$ ) Lax pair (14). This implies explicit formulae for the solutions of the Lagrange top in terms of exponentials and theta functions related to the spectral curve  $C_h$  (see for example Dubrovin [8], E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skiĭ, A. R. Its, V. B. Matveev [5]). Then we note that the Jacobian  $J(C_h)$  of  $C_h$  is just the Lagrange elliptic curve used in the classical theory which provides explicit solutions in terms of exponentials and sigma function related to  $J(C_h)$ .

By performing a unitary operation on the matrix (15) we may put its leading term in diagonal form. Substituting  $a = -m\Omega_3$  in (14) and using the change of variables (25) we obtain the following Lax pair representation for the Lagrange top (2)

$$(29) \quad \left[ A, B - 2i \frac{d}{dt} \right] = 2i \frac{dA}{dt} + [A, B] = 0, \quad \epsilon^2 = i, \quad i^2 = -1$$

where

$$A = A(t, \lambda) = \begin{pmatrix} A_{11}(t, \lambda) & A_{12}(t, \lambda) \\ A_{21}(t, \lambda) & A_{22}(t, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \\ + \begin{pmatrix} (1+m)\Omega_3 & \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) \\ \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) & -(m+1)\Omega_3 \end{pmatrix} \lambda - \begin{pmatrix} \Gamma_3 & \bar{\epsilon}\Gamma_1(t) + \epsilon\Gamma_2(t) \\ \epsilon\Gamma_1(t) + \bar{\epsilon}\Gamma_2(t) & -\Gamma_3 \end{pmatrix}$$

and

$$B = B(t, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \Omega_3 & \bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t) \\ \epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t) & -\Omega_3 \end{pmatrix}.$$

The spectral curve of the above Lax representation is defined by

$$\check{C}_h = \{ \det(A(\lambda) - \mu I) = \mu^2 - f(\lambda) = 0 \},$$

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1.$$

We shall also denote by  $C_h$  the Riemann surface of the compactified affine curve  $\check{C}_h$ . The reader may note the "similarity" between (29) and the Lax pair of the nonlinear Schrödinger equation (for a rigorous statement see Proposition 5.1).

## 3.1 THE BAKER-AKHIEZER FUNCTION

Let us fix a solution  $A(t, \lambda)$  of (29) defined in a neighbourhood of  $t = 0 \in \mathbf{C}$ . We shall also suppose that the point  $P = (\lambda, \mu)$  is such that  $(1, -1)$  is not an eigenvector of the matrix  $A(0, \lambda)$ .

PROPOSITION 3.1. For any  $t \in \mathbf{C}$  in a sufficiently small neighbourhood of the origin, there exists a unique eigenfunction

$$(30) \quad \Psi = \Psi(t, P) = \begin{pmatrix} \Psi^1(t, P) \\ \Psi^2(t, P) \end{pmatrix}, \quad P = (\lambda, \mu) \in \check{C}$$

of  $A(t, \lambda)$  (called the Baker-Akhiezer function) satisfying the conditions

$$(31) \quad 2i \frac{d}{dt} \Psi(t, P) = B(t, \lambda) \Psi(t, P)$$

$$(32) \quad A(t, \lambda) \Psi(t, P) = \mu \Psi(t, P)$$

and normalized by

$$(33) \quad \Psi^1(0, P) + \Psi^2(0, P) = 1.$$

In terms of the coefficients  $A_{ij}(t, \lambda)$  of the matrix  $A = (A_{ij})$  we have

$$(34) \quad \Psi^1(0, P) = \frac{A_{12}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{\mu - A_{22}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}$$

$$(35) \quad \Psi^2(0, P) = \frac{\mu - A_{11}(0, \lambda)}{A_{12}(0, \lambda) + \mu - A_{11}(0, \lambda)} = \frac{A_{21}(0, \lambda)}{A_{21}(0, \lambda) + \mu - A_{22}(0, \lambda)}.$$

*Proof.* Let  $\Phi(t, \lambda)$  be a fundamental matrix for the operator  $B(t, \lambda) - 2i \frac{d}{dt}$  normalized at  $t = 0$ . Then the general solution of (31) is written as

$$(36) \quad \Psi(t, P) = \Phi(t, \lambda) \Psi(0, P), \quad \Phi(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = (\lambda, \mu).$$

As  $A$  and  $B - 2i \frac{d}{dt}$  commute, we have

$$\left( B(t, \lambda) - 2i \frac{d}{dt} \right) A(t, \lambda) \Phi(t, \lambda) = A(t, \lambda) \left( B(t, \lambda) - 2i \frac{d}{dt} \right) \Phi(t, \lambda) = 0$$

and hence  $A(t, \lambda) \Phi(t, \lambda) = \Phi(t, \lambda) M(P)$  for some constant matrix  $M(P)$  computed by substituting  $t = 0$ . Thus  $M(P) = A(0, \lambda)$  and

$$A(0, \lambda) = \Phi^{-1}(t, \lambda) A(t, \lambda) \Phi(t, \lambda).$$

The constants  $\Psi^1(0, P)$ ,  $\Psi^2(0, P)$  are uniquely defined by (32) and (33). Finally,

$$\begin{aligned} A(t, \lambda) \Psi(t, P) &= \Phi(t, \lambda) A(0, \lambda) \Phi^{-1}(t, \lambda) \Phi(t, \lambda) \Psi(0, P) \\ &= \Phi(t, \lambda) \cdot \mu \cdot \Psi(0, P) \\ &= \mu \Psi(t, P). \end{aligned}$$

The formulae (34), (35) follow from (32), (33).  $\square$

Denote by  $\infty^+$  (respectively  $\infty^-$ ) the point on  $C_h - \check{C}_h$  such that in its neighbourhood  $\mu/\lambda^2 \sim +1$  (resp.  $-1$ ).

PROPOSITION 3.2. *There exists  $t_0 > 0$  such that for any fixed  $t \in \mathbf{C}$ ,  $|t| < t_0$ , the Baker-Akhiezer vector-function  $\Psi(t, P)$  is meromorphic in  $P$  on the affine curve  $\check{C}_h$  and has two poles at  $P_1, P_2 \in C_h$  which do not depend on  $t$ . In a neighbourhood of the two infinite points  $\infty^\pm$  on  $C_h$  we have*

$$(37) \quad \Psi^1(t, P) = \begin{cases} (1 + O(\lambda^{-1})) \exp(-\frac{i}{2}(\lambda + \Omega_3)t), & P \rightarrow \infty^+ \\ O(\lambda^{-1}) \exp(+\frac{i}{2}(\lambda + \Omega_3)t), & P \rightarrow \infty^- \end{cases}$$

$$(38) \quad \Psi^2(t, P) = \begin{cases} O(\lambda^{-1}) \exp(-\frac{i}{2}(\lambda + \Omega_3)t), & P \rightarrow \infty^+ \\ (1 + O(\lambda)^{-1}) \exp(+\frac{i}{2}(\lambda + \Omega_3)t), & P \rightarrow \infty^-, \end{cases}$$

where  $i = \sqrt{-1}$ . Moreover,  $\Psi^1(t, P)$  ( $\Psi^2(t, P)$ ) has exactly one zero on  $\check{C}_h$  and the refined asymptotic estimates of  $\Psi^1$  at  $\infty^-$  and of  $\Psi^2$  at  $\infty^+$  read

$$(39) \quad \Psi^1(t, P) = \left[ -\frac{\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)}{2\lambda} + O(\lambda^{-2}) \right] \exp(+\frac{i}{2}(\lambda + \Omega_3)t), \quad P \rightarrow \infty^-$$

$$(40) \quad \Psi^2(t, P) = \left[ +\frac{\epsilon\Omega_1(t) + \bar{\epsilon}\Omega_2(t)}{2\lambda} + O(\lambda^{-2}) \right] \exp(-\frac{i}{2}(\lambda + \Omega_3)t), \quad P \rightarrow \infty^+.$$

*Proof.* According to (32),  $(\Psi^1, \Psi^2) \in \text{Ker}(A - \mu I)$  and hence

$$(41) \quad \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \frac{\mu - \lambda^2 - (1 + m)\Omega_3\lambda + \Gamma_3(t)}{(\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t))\lambda - \bar{\epsilon}\Gamma_1(t) + \epsilon\Gamma_2(t)}.$$

If  $P \rightarrow \infty^+$  then  $\mu - \lambda^2 - (1 + m)\Omega_3\lambda \sim O(1)$  and using (29), (31), (32) and (41) we compute

$$2i \frac{d}{dt} \ln \Psi^1(t, P) = \lambda + \Omega_3 + (\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)) \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \lambda + \Omega_3 + O(\lambda^{-1})$$

and hence

$$\Psi^1(t, P) = (1 + O(\lambda^{-1})) \exp(-\frac{i}{2}(\lambda + \Omega_3)t).$$

In a similar way if  $P \rightarrow \infty^-$  we obtain

$$\Psi^2(t, P) = (1 + O(\lambda^{-1})) \exp(+\frac{i}{2}(\lambda + \Omega_3)t).$$

To compute the remaining asymptotic estimates we use that if  $P \rightarrow \infty^-$  then

$$(42) \quad \frac{\Psi^1(t, P)}{\Psi^2(t, P)} = \frac{A_{12}(t, \lambda)}{\mu - A_{11}(t, \lambda)} = -\frac{\bar{\epsilon}\Omega_1(t) + \epsilon\Omega_2(t)}{2\lambda} + O(\lambda^{-2})$$

and if  $P \rightarrow \infty^+$  then

$$(43) \quad \frac{\Psi^2(t, P)}{\Psi^1(t, P)} = \frac{A_{21}(t, \lambda)}{\mu - A_{22}(t, \lambda)} = \frac{\epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t)}{2\lambda} + O(\lambda^{-2}).$$

To find the poles of  $\Psi(t, P)$  in  $P$  we note that according to the proof of Proposition 3.1 (and with the same notations) we have

$$(44) \quad \Psi(t, P) = \Phi(t, \lambda)\Psi(0, P), \quad \Phi(0, \lambda) = I_2.$$

If  $|t|$  is sufficiently small, the fundamental matrix  $\Phi(t, \lambda)$  has no poles and  $\det \Phi(t, \lambda) \neq 0$ . It follows that the poles of  $\Phi(t, \lambda)$  and  $\Phi(0, \lambda)$  coincide, and we can obtain them by solving the following quadratic equation

$$\det A(0, \lambda) = (A_{11}(0, \lambda) - A_{12}(0, \lambda))^2 = \mu^2$$

(see (29), (34)). One gets two time independent poles  $P_1, P_2 \in \check{C}_h$  of  $\Psi(t, P)$ .

Finally, the meromorphic one-form  $d \ln \Psi^1$  has a simple pole at  $\infty^-$  with residue  $+1$  and is holomorphic in a neighbourhood of  $\infty^+$ . On the other hand  $\Psi^1(t, P)$  has exactly two poles on  $\check{C}_h$  and hence it has one zero on  $\check{C}_h$ . The same arguments hold for  $\Psi^2(t, P)$ .  $\square$

Let  $A_1, A_2, B_1$  be a basis of  $H_1(\check{C}_h, \mathbf{Z})$  as shown in Figure 2 ( $A_1 \circ B_1 = 1$ ), and let  $\omega_1, \omega_2$  be a basis of  $H^0(C, \Omega^1(\infty^+ + \infty^-))$ , normalized by the conditions

$$\left( \int_{A_i} \omega_j \right)_{i,j=1,2} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}.$$

We shall also suppose that  $\omega_1$  is a holomorphic form on the elliptic curve  $C_h$ . Define now the period matrix

$$\Pi = \begin{pmatrix} 2\pi i & 0 & \tau_1 \\ 0 & 2\pi i & \tau_2 \end{pmatrix},$$

where

$$\tau_1 = \int_{B_1} \omega_1, \quad \tau_2 = \int_{B_1} \omega_2, \quad \operatorname{Re}(\tau_1) < 0.$$

Recall that the generalized Jacobian  $J(C_h; \infty^\pm)$  of  $C_h$  relative to the modulus  $m = \infty^+ + \infty^-$  is identified with  $\mathbf{C}^2 / \Lambda$  where  $\Lambda$  is the lattice in  $\mathbf{C}^2$  generated by the columns of  $\Pi$ . Let

$$\theta_{11}(z) = \theta_{11}(z \mid \tau_1) = \sum_{n=-\infty}^{\infty} \exp\left\{ \frac{1}{2} \tau_1 \left(n + \frac{1}{2}\right)^2 + (z + \pi \sqrt{-1}) \left(n + \frac{1}{2}\right) \right\}, \quad z \in \mathbf{C}$$

be the Jacobi theta function with characteristics  $\left[ \frac{1}{2}, \frac{1}{2} \right]$ ,

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z + \tau_1) = -\exp\left(-z - \frac{1}{2} \tau_1\right) \theta_{11}(z).$$

Denote by  $\Omega$  the unique Abelian differential of second kind on  $C_h$  with poles at  $\infty^\pm$ , principal parts  $\pm \frac{i}{2} d\lambda$  where  $P = (\lambda, \mu)$ ,  $i = \sqrt{-1}$ , and normalized by  $\int_{A_1} \Omega = 0$ . Let  $P_0 \in \check{C}_h$  be a fixed initial point,  $c^\pm$ ,  $U$  be the constants defined by

$$(45) \quad \int_{P_0}^P \Omega = \begin{cases} -\frac{i}{2} \lambda + c^- + o(\lambda^{-1}), & P \rightarrow \infty^+ \\ +\frac{i}{2} \lambda + c^+ + o(\lambda^{-1}), & P \rightarrow \infty^- \end{cases}, \quad \int_{B_1} \Omega = U.$$

Define the Abel-Jacobi map

$$A: \text{Div}^0(C_h) \rightarrow J(C_h) : \sum P_i - \sum Q_i \mapsto \int_{\Sigma}^{\sum P_i} \omega_1.$$

Here, and henceforth, we make the convention that the paths of integration between divisors are taken within  $C_h$  cut along its homology basis  $A_1, B_1$ , which we assume does not contain points of these divisors.

PROPOSITION 3.3. *The Baker-Akhiezer function is explicitly given by*

$$(46) \quad \Psi^1(t, P) = \text{const}_1 \cdot \exp \left[ t \left( \int_{P_0}^P \Omega - c^- - \frac{i}{2} \Omega_3 \right) \right] \frac{\theta_{11}(\mathcal{A}(P + \infty^- - P_1 - P_2) + tU)}{\theta_{11}(\mathcal{A}(\infty^+ + \infty^- - P_1 - P_2) + tU)}$$

$$(47) \quad \Psi^2(t, P) = \text{const}_2 \cdot \exp \left[ t \left( \int_{P_0}^P \Omega - c^+ + \frac{i}{2} \Omega_3 \right) \right] \frac{\theta_{11}(\mathcal{A}(P + \infty^+ - P_1 - P_2) + tU)}{\theta_{11}(\mathcal{A}(\infty^+ + \infty^- - P_1 - P_2) + tU)}$$

where

$$\text{const}_1 = \frac{\theta_{11}(\mathcal{A}(P - \infty^-))}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(P - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(P - P_2))}$$

$$\text{const}_2 = \frac{\theta_{11}(\mathcal{A}(P - \infty^+))}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(P - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(P - P_2))}$$

and  $P_1, P_2$  are the poles of  $\Psi$ .

The proof of the above proposition is based on a general fact: the properties of  $\Psi$  enumerated in Proposition 3.2 define it uniquely. Indeed, if  $\Psi$  and  $\tilde{\Psi}$  are vector functions both satisfying the assumptions of Proposition 3.2, then the functions  $\Psi^1$  and  $\tilde{\Psi}^1$  (resp.  $\Psi^2$  and  $\tilde{\Psi}^2$ ) meromorphic on  $C_h$  have the same poles. Using this and the asymptotic estimates at infinity we conclude that  $\Psi^1/\tilde{\Psi}^1$  and  $\Psi^2/\tilde{\Psi}^2$  are meromorphic functions on  $C_h$  which have one pole (at  $\tilde{\Psi}^i = 0$ ). Moreover

$$\Psi_1(t, \infty^-)/\tilde{\Psi}_1(t, \infty^-) = 1, \quad \Psi_2(t, \infty^-)/\tilde{\Psi}_2(t, \infty^-) = 1$$

and hence  $\Psi = \tilde{\Psi}$ . Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1.  $\square$

### 3.2 SOLUTIONS OF THE LAGRANGE TOP

Let  $z = (z_1, z_2) \in J(C_h; \infty^\pm)$ . It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on  $J(C_h; \infty^\pm)$ . We shall see that they give solutions of the Lagrange top. By (16) we compute that  $\frac{d}{dt}z = \text{constant}$ , where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \quad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \quad a = -m\Omega_3.$$

**THEOREM 3.4.** *The following equations hold*

$$(48) \quad \bar{\epsilon} \Omega_1(t) + \epsilon \Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2},$$

$$(49) \quad \epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2},$$

where

$$(50) \quad \begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned}$$

and

$$\text{const}_3 = \frac{2i V_1 \theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(\infty^- - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(\infty^- - P_2))},$$

$$\text{const}_4 = \frac{2i V_1 \theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(\infty^+ - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}.$$