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and hence  $\Psi = \tilde{\Psi}$ . Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1.  $\square$

### 3.2 SOLUTIONS OF THE LAGRANGE TOP

Let  $z = (z_1, z_2) \in J(C_h; \infty^\pm)$ . It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on  $J(C_h; \infty^\pm)$ . We shall see that they give solutions of the Lagrange top. By (16) we compute that  $\frac{d}{dt}z = \text{constant}$ , where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \quad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \quad a = -m\Omega_3.$$

**THEOREM 3.4.** *The following equations hold*

$$(48) \quad \bar{\epsilon} \Omega_1(t) + \epsilon \Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2},$$

$$(49) \quad \epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2},$$

where

$$(50) \quad \begin{aligned} z_2 &= tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \\ \tau_2 &= \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2 \end{aligned}$$

and

$$\text{const}_3 = \frac{2i V_1 \theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^- - \infty^+))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_1))}{\theta_{11}(\mathcal{A}(\infty^- - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}{\theta_{11}(\mathcal{A}(\infty^- - P_2))},$$

$$\text{const}_4 = \frac{2i V_1 \theta'_{11}(0)}{\theta_{11}(\mathcal{A}(\infty^+ - \infty^-))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_1))}{\theta_{11}(\mathcal{A}(\infty^+ - P_1))} \cdot \frac{\theta_{11}(\mathcal{A}(\infty^- - P_2))}{\theta_{11}(\mathcal{A}(\infty^+ - P_2))}.$$

Let us denote

$$\begin{aligned}\omega_1 &= \pm(\omega_1^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm, \\ \omega_2 &= \pm(\omega_2^1 \lambda + \omega_2^0 + O(\lambda^{-1})) d(\lambda^{-1}), & P = (\lambda, \mu) \rightarrow \infty^\pm.\end{aligned}$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\begin{aligned}\omega_1^0 &= -i \int_{B_1} \Omega = -iV_1, & \omega_2^0 &= i(c^+ - c^-), \\ V_2 &= -c^+ + c^- + i\Omega_3, & \mathcal{A}(\infty^+ - \infty^-) &= \int_{B_1} \omega_2.\end{aligned}$$

*Proof.* The identity  $\omega_1^0 = -i \int_{B_1} \Omega$  is a reciprocity law between the differential of the first kind  $\omega_1$  and the differential of the second kind  $\Omega$  [13]. It is obtained by integrating  $\pi(P)\omega_1$ , where  $\pi(P) = \int_{P_0}^P \Omega$ , along the border of  $C_h$  cut along its homology basis  $A_1, B_1$ . On the other hand

$$\omega_1 = 2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i \left( \int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} = -iV_1.$$

Similarly the identity  $\omega_2^0 = i(c^+ - c^-)$  is a reciprocity law between the differential of the third kind  $\omega_2$  and the differential of the second kind  $\Omega$ , and  $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2$  is a reciprocity law between the differential of the third kind  $\omega_2$  and the differential of the first kind  $\omega_1$ . Finally, as

$$\omega_2 = \frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} \frac{d\lambda}{\mu} - \frac{\lambda d\lambda}{\mu} \quad \text{we have} \quad \omega_2^0 = -\frac{\int_{A_1} \frac{\lambda d\lambda}{\mu}}{\int_{A_1} \frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$$

and hence  $V_2 = -c^+ + c^- + i\Omega_3$ .  $\square$

*Proof of Theorem 3.4.* According to (42), (43)

$$\bar{\epsilon} \Omega_1(t) + \epsilon \Omega_2(t) = -2 \lim_{P \rightarrow \infty^-} \frac{\lambda \Psi^1(t, P)}{\Psi^2(t, P)}$$

and

$$\epsilon \Omega_1(t) + \bar{\epsilon} \Omega_2(t) = +2 \lim_{P \rightarrow \infty^+} \frac{\lambda \Psi^2(t, P)}{\Psi^1(t, P)}.$$

To compute the limit we use (46), (47) and

$$\lim_{P \rightarrow \infty^-} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^-)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

$$\lim_{P \rightarrow \infty^+} \lambda(P) \theta_{11}(\mathcal{A}(P - \infty^+)) = \theta'_{11}(0) \frac{d}{ds} \Big|_{s=0} \int^s \omega_1 = \omega_1^0 \theta'_{11}(0)$$

(see Lemma 3.5).  $\square$

### 3.3 EFFECTIVIZATION

Let  $\wp, \zeta, \sigma$  be the Weierstrass functions related to the elliptic curve  $\Gamma$  defined by

$$(51) \quad \eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve  $C$  with affine equation

$$(52) \quad \mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution  $(\lambda, \mu) \rightarrow (\bar{\lambda}, \bar{\mu})$ , and put

$$(53) \quad g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \quad g_3 = \det \begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves  $C$  and  $\Gamma$  are isomorphic over  $\mathbf{C}$  and that under this isomorphism

$$(54) \quad \frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call  $\Gamma$  the Jacobian  $J(C)$  of the elliptic curve  $C$  and we write  $J(C) = \Gamma$ . Note that  $J(C)$  and  $\Gamma$  are real isomorphic and that  $J(C)$  and  $C$  are not real isomorphic.

Further we make the substitution (23) and  $C$  becomes the spectral curve  $\tilde{C}_h$  of Adler and van Moerbeke  $\{\mu^2 + f(\lambda) = 0\}$ , where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + (2h_3 + m(m+1)h_4^2)\lambda^2 - 2h_2\lambda + 1$$

and  $\Gamma$  becomes the Lagrange curve  $\Gamma_h$ . Recall that, as we explained at the end of Section 2, the curve  $C_h$  with an equation  $\{\mu^2 = f(\lambda)\}$  and antiholomorphic involution  $(\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$ , is isomorphic over  $\mathbf{R}$  to  $\tilde{C}_h$ , so we write  $C_h = \tilde{C}_h$ . The Jacobian curve  $J(C_h) = \Gamma_h$  was computed by