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and hence $\Psi = \widetilde{\Psi}$. Finally, the reader may check that the functions (46) and (47) have the analyticity properties from Proposition 3.2 and hence they coincide with the Baker-Akhiezer function defined in Proposition 3.1. □

3.2 Solutions of the Lagrange top

Let $z = (z_1, z_2) \in J(C_h; \infty^\pm)$. It is easy to check that the functions

$$\frac{\theta_{11}(z_1 \pm \tau_2)}{\theta_{11}(z_1)} e^{\mp z_2}$$

live on $J(C_h; \infty^\pm)$. We shall see that they give solutions of the Lagrange top. By (16) we compute that $\frac{d}{dt} z = \text{constant}$, where

$$\frac{dz}{dt} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 2\pi i \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} & \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda d\lambda}{\mu} & \int_{A_2} \frac{\lambda d\lambda}{\mu} \end{pmatrix}^{-1} \begin{pmatrix} -i \\ -ai \end{pmatrix},$$

$$\int_{A_2} \frac{d\lambda}{\mu} = 0, \qquad \int_{A_2} \frac{\lambda d\lambda}{\mu} = -2\pi i$$

so

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \left(\int_{A_1} \frac{d\lambda}{\mu} \right)^{-1} \begin{pmatrix} 2\pi \\ -i \int_{A_1} \frac{\lambda d\lambda}{\mu} + ai \int_{A_1} \frac{d\lambda}{\mu} \end{pmatrix}, \qquad a = -m\Omega_3 .$$

THEOREM 3.4. *The following equations hold*

(48) $$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = \text{const}_3 \frac{\theta_{11}(z_1 - \tau_2)}{\theta_{11}(z_1)} e^{-z_2},$$

(49) $$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = \text{const}_4 \frac{\theta_{11}(z_1 + \tau_2)}{\theta_{11}(z_1)} e^{+z_2},$$

where

(50) $$z_2 = tV_2, \quad z_1 = tV_1 + \mathcal{A}(\infty^+ + \infty^- - P_1 - P_2), \qquad \tau_2 = \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1} \omega_2$$

and

$$\text{const}_3 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^- - \infty^+)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)},$$

$$\text{const}_4 = \frac{2i\, V_1\, \theta'_{11}(0)}{\theta_{11}\big(\mathcal{A}(\infty^+ - \infty^-)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_1)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_1)\big)} \cdot \frac{\theta_{11}\big(\mathcal{A}(\infty^- - P_2)\big)}{\theta_{11}\big(\mathcal{A}(\infty^+ - P_2)\big)}.$$

Let us denote

$$\omega_1 = \pm\big(\omega_1^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm},$$
$$\omega_2 = \pm\big(\omega_2^1\lambda + \omega_2^0 + O(\lambda^{-1})\big)\, d\big(\lambda^{-1}\big), \qquad P = (\lambda, \mu) \to \infty^{\pm}.$$

To prove Theorem 3.4 we shall need the following

LEMMA 3.5. *The above defined differentials are such that*

$$\omega_1^0 = -i\int_{B_1}\Omega = -iV_1\,, \qquad \omega_2^0 = i\,(c^+ - c^-)\,,$$
$$V_2 = -c^+ + c^- + i\Omega_3\,, \qquad \mathcal{A}(\infty^+ - \infty^-) = \int_{B_1}\omega_2\,.$$

Proof. The identity $\omega_1^0 = -i\int_{B_1}\Omega$ is a reciprocity law between the differential of the first kind ω_1 and the differential of the second kind Ω [13]. It is obtained by integrating $\pi(P)\omega_1$, where $\pi(P) = \int_{P_0}^{P}\Omega$, along the border of C_h cut along its homology basis A_1, B_1. On the other hand

$$\omega_1 = 2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1}\frac{d\lambda}{\mu}$$

and hence

$$\omega_1^0 = -2\pi i\left(\int_{A_1}\frac{d\lambda}{\mu}\right)^{-1} = -iV_1\,.$$

Similarly the identity $\omega_2^0 = i(c^+ - c^-)$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the second kind Ω, and $\mathcal{A}(\infty^+ - \infty^-) = \int_{B_1}\omega_2$ is a reciprocity law between the differential of the third kind ω_2 and the differential of the first kind ω_1. Finally, as $\omega_2 = \frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}}\frac{d\lambda}{\mu} - \frac{\lambda\, d\lambda}{\mu}$ we have $\omega_2^0 = -\frac{\int_{A_1}\frac{\lambda d\lambda}{\mu}}{\int_{A_1}\frac{d\lambda}{\mu}} - (1+m)\Omega_3 = -iV_1 - \Omega_3$ and hence $V_2 = -c^+ + c^- + i\Omega_3$. □

Proof of Theorem 3.4. According to (42), (43)

$$\bar{\epsilon}\,\Omega_1(t) + \epsilon\,\Omega_2(t) = -2\lim_{P\to\infty^-}\frac{\lambda\Psi^1(t,P)}{\Psi^2(t,P)}$$

and

$$\epsilon\,\Omega_1(t) + \bar{\epsilon}\,\Omega_2(t) = +2\lim_{P\to\infty^+}\frac{\lambda\Psi^2(t,P)}{\Psi^1(t,P)}\,.$$

To compute the limit we use (46), (47) and

$$\lim_{P\to\infty^-} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^-)\big) = \theta'_{11}(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta'_{11}(0)$$

$$\lim_{P\to\infty^+} \lambda(P)\,\theta_{11}\big(\mathcal{A}(P-\infty^+)\big) = \theta'_{11}(0)\,\frac{d}{ds}\Big|_{s=0}\int^s \omega_1 = \omega_1^0\,\theta'_{11}(0)$$

(see Lemma 3.5). □

3.3 Effectivization

Let $\wp, \zeta, \sigma$ be the Weierstrass functions related to the elliptic curve Γ defined by

(51) $$\eta^2 = 4\xi^3 - g_2\xi - g_3$$

(we use the standard notations of [4]).

Consider also the *real* elliptic curve C with affine equation

(52) $$\mu^2 + \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

and natural anti-holomorphic involution $(\lambda,\mu) \to \left(\overline{\lambda},\overline{\mu}\right)$, and put

(53) $$g_2 = a_4 + 3\left(\frac{a_2}{6}\right)^4 - 4\frac{a_1}{4}\frac{a_3}{4}, \qquad g_3 = \det\begin{pmatrix} 1 & \frac{a_1}{4} & \frac{a_2}{6} \\ \frac{a_1}{4} & \frac{a_2}{6} & \frac{a_3}{4} \\ \frac{a_2}{6} & \frac{a_3}{4} & a_4 \end{pmatrix}.$$

It is well known that the curves C and Γ are isomorphic over $\mathbf{C}$ and that under this isomorphism

(54) $$\frac{d\lambda}{\mu} = \frac{d\xi}{\eta}.$$

Following Weil [25] we call Γ the Jacobian $J(C)$ of the elliptic curve C and we write $J(C) = \Gamma$. Note that $J(C)$ and Γ are real isomorphic and that $J(C)$ and C are not real isomorphic.

Further we make the substitution (23) and C becomes the spectral curve $\widetilde{C}_h$ of Adler and van Moerbeke $\left\{\mu^2 + f(\lambda) = 0\right\}$, where

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1$$

and Γ becomes the Lagrange curve Γ_h. Recall that, as we explained at the end of Section 2, the curve C_h with an equation $\{\mu^2 = f(\lambda)\}$ and antiholomorphic involution $(\lambda,\mu) \to (\overline{\lambda}, -\overline{\mu})$, is isomorphic over $\mathbf{R}$ to $\widetilde{C}_h$, so we write $C_h = \widetilde{C}_h$. The Jacobian curve $J(C_h) = \Gamma_h$ was computed by