

**Zeitschrift:** L'Enseignement Mathématique  
**Band:** 44 (1998)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE COMPLEX GEOMETRY OF THE LAGRANGE TOP  
**Kapitel:** 5. The Lagrange top and the non-linear Schrödinger equation  
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**DOI:** <https://doi.org/10.5169/seals-63901>

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## 5. THE LAGRANGE TOP AND THE NON-LINEAR SCHRÖDINGER EQUATION

Our final remark concerns a previously unknown relation between the real solutions of the Lagrange top and the one-gap solutions of the nonlinear Schrödinger equation

$$(NLS^\pm) \quad u_{xx} = iu_t \pm 2|u|^2u.$$

In the physical applications both forms of  $(NLS)$  are of interest. Comparing Theorem 2.2 to the results of Previato [20] we note that the invariant manifolds of one-gap solutions of the  $NLS$  equation are isomorphic to the invariant manifolds of the Lagrange top. This relation can be made explicit if we compare the expressions for the solutions found in Theorem 3.4 to the well known formulae for  $u(x, t)$  (cf. [5, 20]). We shall see that the  $S^\pm$ -real solutions of the Lagrange top give also one-gap solutions of  $NLS^\pm$  equation. Recall that, according to the preceding section, an  $S^-$ -real solution is a usual real solution of the Lagrange top (2), and that an  $S^+$ -real solution is a real solution of the system (60).

Let  $X_E, X_{\Omega_3}$  be the Hamiltonian vector fields (2) and (3) respectively and put

$$\frac{\partial}{\partial x} = \frac{1}{2}X_E, \quad \frac{\partial}{\partial t} = \frac{1}{4}(m-1)\Omega_3 X_E + \frac{1}{8}(2h_3 - (3m+1)\Omega_3^2)X_{\Omega_3}.$$

As  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial t}$  define translation invariant vector fields on the generalized Jacobian  $J(C_h; \infty^\pm)$  then fixing an arbitrary point for origin we may introduce  $(x, t)$  coordinates on  $J(C_h; \infty^\pm)$  (and hence on the complex invariant manifold  $T_h$ ). If the real part  $T_h^{\mathbf{R}}$  of  $T_h$  is not empty, then we shall choose for origin a real point. As the real vector fields  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial t}$  are tangent to the Liouville torus  $T_h^{\mathbf{R}}$ , then  $(x, t)$  provide real affine coordinates on it. Denote, lastly, by  $u^-(x, t)$  the restriction of the function  $\bar{\epsilon}\Omega_1 + \epsilon\Omega_2$  on the Liouville torus  $T_h^{\mathbf{R}}$  of the Lagrange top (2).

Similarly, let  $u^+(x, t)$  be the restriction of the function  $\bar{\epsilon}\Omega_1 + \epsilon\Omega_2$  on a connected component of the  $S^+$ -real part of  $J(C_h; \infty^\pm)$ . If the origin belongs to this component too, then as above we conclude that  $x, t \in \mathbf{R}$ .

**PROPOSITION 5.1.** *The functions  $u^+(x, t)$  and  $u^-(x, t)$  satisfy  $NLS^+$  and  $NLS^-$  respectively.*

The proof of the above Proposition is a straightforward computation (compare with [20], Theorem 2.2). From the definition of  $u^\pm$  we get

$\bar{u}^- = \bar{\epsilon}\Omega_2 + \epsilon\Omega_1$  and  $\bar{u}^+ = -\bar{\epsilon}\Omega_2 - \epsilon\Omega_1$ . It follows that  $|u^\pm|^2 = \mp(\Omega_1^2 + \Omega_2^2)$  and it is easy to check that

$$u_{xx}^\pm = iu_t^\pm \pm 2|u^\pm|^2 u^\pm$$

is equivalent to the system

$$\begin{aligned} (\Omega_1)_{xx} + (\Omega_2)_t &= \pm 2\Omega_1(\Omega_1^2 + \Omega_2^2) \\ (\Omega_2)_{xx} - (\Omega_1)_t &= \pm 2\Omega_2(\Omega_1^2 + \Omega_2^2) \end{aligned}$$

where  $\Omega_1, \Omega_2$  are defined on the  $S^\pm$ -real part of  $T_h$  respectively. Using (2) we get for the derivatives along  $X_E$

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 = -m\Omega_1\Omega_3^2 - \Omega_1\Gamma_3$$

and as

$$\Gamma_3 = \frac{1}{2}(\Omega_1^2 + \Omega_2^2 + (1+m)\Omega_3^2) - E,$$

then

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 = -\frac{1}{2}\Omega_1(\Omega_1^2 + \Omega_2^2) + \Omega_1(E - \frac{3m+1}{2}\Omega_3^2).$$

Finally, as  $X_{\Omega_3}\Omega_2 = -\Omega_1$  we conclude that

$$\begin{aligned} (\Omega_1)_{xx} + (\Omega_2)_t &= -2\Omega_1(\Omega_1^2 + \Omega_2^2) \\ (\Omega_2)_{xx} - (\Omega_1)_t &= -2\Omega_2(\Omega_1^2 + \Omega_2^2). \end{aligned}$$

This proves also that  $u^+$  is a solution of  $NLS^+$  (we just have to substitute  $\Omega_1 \mapsto i\Omega_1, \Omega_2 \mapsto i\Omega_2$ ).  $\square$