Zeitschrift: L'Enseignement Mathématique

Band: 44 (1998)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE COMPLEX GEOMETRY OF THE LAGRANGE TOP

Kapitel: 5. The Lagrange top and the non-linear Schrödinger equation

Autor: Gavrilov, Lubomir / ZHIVKOV, Angel

DOI: https://doi.org/10.5169/seals-63901

5. THE LAGRANGE TOP AND THE NON-LINEAR SCHRÖDINGER EQUATION

Our final remark concerns a previously unknown relation between the real solutions of the Lagrange top and the one-gap solutions of the nonlinear Schrödinger equation

$$(NLS^{\pm}) \qquad u_{xx} = iu_t \pm 2|u|^2 u\,.$$

In the physical applications both forms of (NLS) are of interest. Comparing Theorem 2.2 to the results of Previato [20] we note that the invariant manifolds of one-gap solutions of the NLS equation are isomorphic to the invariant manifolds of the Lagrange top. This relation can be made explicit if we compare the expressions for the solutions found in Theorem 3.4 to the well known formulae for $u(x, t)$ (cf. [5, 20]). We shall see that the $S^{\pm}$-real solutions of the Lagrange top give also one-gap solutions of $NLS^{\pm}$ equation. Recall that, according to the preceding section, an S^{-}-real solution is a usual real solution of the Lagrange top (2), and that an S^{+}-real solution is a real solution of the system (60).

Let X_E, X_{Ω_3} be the Hamiltonian vector fields (2) and (3) respectively and put

$$\frac{\partial}{\partial x} = \tfrac{1}{2}X_E, \qquad \frac{\partial}{\partial t} = \tfrac{1}{4}(m-1)\Omega_3 X_E + \tfrac{1}{8}\left(2h_3 - (3m+1)\Omega_3^2\right)X_{\Omega_3}\,.$$

As $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ define translation invariant vector fields on the generalized Jacobian $J(C_h; \infty^{\pm})$ then fixing an arbitrary point for origin we may introduce (x, t) coordinates on $J(C_h; \infty^{\pm})$ (and hence on the complex invariant manifold T_h). If the real part $T_h^{\mathbf{R}}$ of T_h is not empty, then we shall choose for origin a real point. As the real vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ are tangent to the Liouville torus $T_h^{\mathbf{R}}$, then (x, t) provide real affine coordinates on it. Denote, lastly, by $u^{-}(x, t)$ the restriction of the function $\bar{\epsilon}\,\Omega_1 + \epsilon\,\Omega_2$ on the Liouville torus $T_h^{\mathbf{R}}$ of the Lagrange top (2).

Similarly, let $u^{+}(x, t)$ be the restriction of the function $\bar{\epsilon}\,\Omega_1 + \epsilon\,\Omega_2$ on a connected component of the S^{+}-real part of $J(C_h; \infty^{\pm})$. If the origin belongs to this component too, then as above we conclude that $x, t \in \mathbf{R}$.

PROPOSITION 5.1. *The functions $u^{+}(x, t)$ and $u^{-}(x, t)$ satisfy NLS^{+} and NLS^{-} respectively.*

The proof of the above Proposition is a straightforward computation (compare with [20], Theorem 2.2). From the definition of $u^{\pm}$ we get

$\bar{u}^- = \bar{\epsilon}\,\Omega_2 + \epsilon\,\Omega_1$ and $\bar{u}^+ = -\bar{\epsilon}\,\Omega_2 - \epsilon\,\Omega_1$. It follows that $|u^\pm|^2 = \mp(\Omega_1^2 + \Omega_2^2)$ and it is easy to check that

$$u^\pm_{xx} = iu^\pm_t \pm 2|u^\pm|^2 u^\pm$$

is equivalent to the system

$$(\Omega_1)_{xx} + (\Omega_2)_t = \pm 2\Omega_1(\Omega_1^2 + \Omega_2^2)$$
$$(\Omega_2)_{xx} - (\Omega_1)_t = \pm 2\Omega_2(\Omega_1^2 + \Omega_2^2)$$

where Ω_1, Ω_2 are defined on the $S^\pm$-real part of T_h respectively. Using (2) we get for the derivatives along X_E

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 \;\; = -m\Omega_1\Omega_3^2 - \Omega_1\Gamma_3$$

and as

$$\Gamma_3 = \tfrac{1}{2}\left(\Omega_1^2 + \Omega_2^2 + (1+m)\Omega_3^2\right) - E\,,$$

then

$$\ddot{\Omega}_1 + (m-1)\Omega_3\dot{\Omega}_2 = -\tfrac{1}{2}\Omega_1(\Omega_1^2 + \Omega_2^2) + \Omega_1\left(E - \tfrac{3m+1}{2}\Omega_3^2\right).$$

Finally, as $X_{\Omega_3}\Omega_2 = -\Omega_1$ we conclude that

$$(\Omega_1)_{xx} + (\Omega_2)_t = -2\Omega_1(\Omega_1^2 + \Omega_2^2)$$
$$(\Omega_2)_{xx} - (\Omega_1)_t = -2\Omega_2(\Omega_1^2 + \Omega_2^2)\,.$$

This proves also that u^+ is a solution of NLS^+ (we just have to substitute $\Omega_1 \mapsto i\Omega_1$, $\Omega_2 \mapsto i\Omega_2$). $\square$