1. A THEOREM OF GROTHENDIECK

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The following theorem is a special case of Grothendieck's theorems, and the proof can be found in [Mu] §5, [H] §3.12, or [EGA] III, §7.7.5, 7.9.4.

THEOREM 1.1. *Let* $q\colon V \to T$ *be a proper flat morphism of noetherian schemes and let* $\mathcal{L}$ *be an invertible sheaf on* V. *For each* $t \in T$ *denote the fiber* $V \otimes_T \operatorname{spec}(k(t))$ *of* q *at* t *by* V_t, *where* $k(t)$ *is the residue field of* T *at* t. *Denote the inverse image of* $\mathcal{L}$ *on* V_t *by* $\mathcal{L}_t$.

(a) *The function* $t \mapsto \chi(\mathcal{L}_t) = \sum_i (-1)^i \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *is locally constant on* T.

(b) *For each* i, *the function* $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *on* T *is upper semicontinuous.*

(c) *If* T *is reduced and connected and if* $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ *is a constant function on* T, *then* $R^i q_* \mathcal{L}$ *is a locally free sheaf on* T *and the map* $R^i q_* \mathcal{L} \otimes_{\mathcal{O}_T} k(t) \to H^i(V_t, \mathcal{L}_t)$ *is an isomorphism.*

(d) *If* $H^1(V_t, \mathcal{L}_t) = 0$ *for all* $t \in T$, *then* $R^1 q_* \mathcal{L} = 0$ *and* $q_* \mathcal{L}$ *is a locally free sheaf. Moreover the formation of* $q_* \mathcal{L}$ *commutes with any base change.*

2. RELATIVE EFFECTIVE CARTIER DIVISORS

Let $q\colon X \to T$ be a morphism of noetherian schemes. A *relative effective Cartier divisor* on X/T is an effective Cartier divisor on X that is flat over T when regarded as a closed subscheme of X. When $T = \operatorname{spec}(R)$ is affine, a closed subscheme D of X is a relative effective Cartier divisor if and only if there exists an open affine covering $U_i = \operatorname{spec}(R_i)$ of X and $g_i \in R_i$ such that

(a) $D \cap U_i = \operatorname{spec}\big(R_i/(g_i)\big)$;

(b) g_i is not a zero divisor;

(c) $R_i/(g_i)$ is flat over R.

REMARK 2.1. Let D be an effective Cartier divisor on X/T, let $\mathcal{I}(D)$ be the sheaf of ideals defining D, and let $\mathcal{L}(D)$ be the invertible sheaf corresponding to D. We have $\mathcal{L}(D) = \mathcal{I}(D)^{-1}$. The inclusion $\mathcal{I}(D) \subset \mathcal{O}_X$ induces $\mathcal{O}_X \subset \mathcal{I}(D)^{-1} = \mathcal{L}(D)$, hence a section s_D of $\mathcal{L}(D)$.