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**Autor:** van Dijk, G. / PASQUALE, A.  
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In Section 6 we prove that the  $\tau$ -Abel transform is an isomorphism of  $\mathcal{D}(G; \chi_\tau)$  onto the convolution algebra  $\mathcal{D}_+(\mathbf{R})$  of the even  $C^\infty$  compactly supported functions on  $\mathbf{R}$ . The inversion formula is explicitly written. The Paley-Wiener Theorem for the  $\tau$ -spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the  $\tau$ -spherical transform.

Similar results for  $SU(n, 1)$  have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

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### 1. THE FINE STRUCTURE OF $Sp(1, n)$

Let  $\mathbf{H}$  be the skew-field of the quaternions. Consider on the right  $\mathbf{H}$ -vector space  $\mathbf{H}^{n+1}$  the Hermitian form

$$(1.1) \quad [x, y] = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n,$$

the bar sign denoting quaternionic conjugation: if  $1, i, j, k$  are the quaternionic units and  $q = a + ib + jc + kd \in \mathbf{H}$  (with  $a, b, c, d \in \mathbf{R}$ ), then  $\bar{q} = a - ib - jc - kd$ . Let  $G = Sp(1, n)$  be the group  $U(1, n; \mathbf{H})$  of  $(n+1) \times (n+1)$  matrices with coefficients in  $\mathbf{H}$  which preserve this form. For  $n = 1$ ,  $G$  is called the De Sitter group. Let  $Sp(m)$  indicate the group  $U(m; \mathbf{H})$  of  $m \times m$  matrices with coefficients in  $\mathbf{H}$  which preserve the inner product  $(x, y) = \bar{y}_1 x_1 + \cdots + \bar{y}_m x_m$  of  $\mathbf{H}^m$ . In particular,  $Sp(1)$  consists of the quaternions  $q = a + ib + jc + kd$  with norm  $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$  equal to 1.  $Sp(1)$  is canonically isomorphic to  $SU(2)$ . The group  $G$  acts on the projective space  $P_n(\mathbf{H})$ . Let  $\Omega$  denote the image of the open set  $\{x \in \mathbf{H}^{n+1} : [x, x] > 0\}$  under the canonical map  $\mathbf{H}^{n+1} \setminus \{0\} \rightarrow P_n(\mathbf{H})$ . Then  $G$  acts transitively on  $\Omega$ , and the stabilizer of the quaternionic line generated by the vector  $(1, 0, \dots, 0)$  is the group

$$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & U \end{bmatrix} : u \in Sp(1), U \in Sp(n) \right\} \equiv Sp(1) \times Sp(n).$$

The homogeneous space  $G/K$  is called the hyperbolic quaternionic space.  $K$  is a maximally compact subgroup of  $G$ .  $G$  is connected and simply connected.

To study the fine structure of  $G$ , we consider its Lie algebra  $\mathfrak{g} = \mathfrak{sp}(1, n)$ . Let  $J$  be the  $(n+1) \times (n+1)$  matrix  $\mathrm{diag}(-1, 1, \dots, 1)$ . For any matrix  $X$  of type  $(n+1, n+1)$  with coefficients in  $\mathbf{H}$  we set  $X^* = J\bar{X}^t J$ , the symbol  $^t$  denoting transposition.

The Lie algebra  $\mathfrak{g}$  consists of the matrices  $X$  which verify the relation

$$X + X^* = 0.$$

These are the matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ \bar{Z}_2^t & Z_3 \end{bmatrix}$$

with  $Z_1$  and  $Z_3$  anti-Hermitian of type  $(1, 1)$  and  $(n, n)$ , respectively, and  $Z_2$  arbitrary. Let  $\theta$  be the anti-involutive automorphism of  $\mathfrak{g}$  defined by

$$\theta X = JXJ.$$

Then  $\theta$  is a Cartan involution with the usual decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . Here  $\mathfrak{k}$  is the Lie algebra of  $K$ . Let  $L$  be the following element of  $\mathfrak{g}$ :

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then  $L \in \mathfrak{p}$  and  $\mathfrak{a} = \mathbf{R}L$  is a maximal Abelian subspace of  $\mathfrak{p}$ . We are going to diagonalize  $\mathrm{ad}L$ . The centralizer of  $L$  in  $\mathfrak{k}$  is the subalgebra  $\mathfrak{m}$  of  $\mathfrak{g}$  of the matrices

$$\begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with  $u \in \mathbf{H}$ ,  $u + \bar{u} = 0$  and  $V$  a matrix of type  $(n-1, n-1)$  satisfying  $V + \bar{V}^t = \mathbf{0}$ . The non-zero eigenvalues of  $\mathrm{ad}L$  are  $\alpha = 1$ ,  $-\alpha$ ,  $\pm 2\alpha$ . The space  $\mathfrak{g}_\alpha$  consists of the matrices

$$X = \begin{bmatrix} 0 & z^* & 0 \\ z & \mathbf{0} & -z \\ 0 & z^* & 0 \end{bmatrix}$$

where  $z$  is a matrix of type  $(n-1, 1)$  with coefficients in  $\mathbf{H}$ , and  $z^* := \bar{z}^t$ . The real dimension of  $\mathfrak{g}_\alpha$  is  $m_\alpha = 4(n-1)$ . The space  $\mathfrak{g}_{2\alpha}$  consists of the matrices of the form

$$X = \begin{bmatrix} w & 0 & -w \\ 0 & \mathbf{0} & 0 \\ w & 0 & -w \end{bmatrix}$$

with  $w \in \mathbf{H}$ ,  $w + \bar{w} = 0$ . The dimension of  $\mathfrak{g}_{2\alpha}$  is equal to  $m_{2\alpha} = 3$ . We have  $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$ .

Let  $A$  be the subgroup  $\exp \mathfrak{a}$ . This is the subgroup of the matrices

$$a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

where  $t$  is a real number. The centralizer of  $A$  in  $K$  is the subgroup  $M$  of the matrices

$$m(u, V) = \begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with  $u \in \text{Sp}(1)$  and  $V \in \text{Sp}(n-1)$ . The Lie algebra of  $M$  is  $\mathfrak{m}$ . The subspace  $\mathfrak{n} = \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$  is a (real) nilpotent subalgebra. Set  $N = \exp \mathfrak{n}$ . This is the subgroup of the matrices

$$n(w, z) = \begin{bmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{bmatrix}$$

where  $w \in \mathbf{H}$  satisfies  $w + \bar{w} = 0$  and  $z = [z_1, \dots, z_{n-1}]^t$  is a matrix of type  $(n-1, 1)$  with coefficients in  $\mathbf{H}$ . We have set  $z^* = \bar{z}^t$  and  $[z, z] = -\bar{z}_1 z_1 - \dots - \bar{z}_{n-1} z_{n-1}$ .

The composition law in  $N$  is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \Im[z, z'], z + z'),$$

where  $\Im q := \frac{q - \bar{q}}{2}$  for  $q \in \mathbf{H}$ . The subgroups  $A$  and  $M$  normalize  $N$ :

$$\begin{aligned} a_t n(w, z) a_{-t} &= n(e^{2t} w, e^t z), \\ m(u, V) n(w, z) m(u, V)^{-1} &= n(uw\bar{u}, Vz\bar{u}). \end{aligned}$$

Let  $2\rho$  be the trace of the restriction of  $\text{ad} L$  to  $\mathfrak{n}$ :

$$(1.2) \quad \rho = \frac{1}{2}(m_{\alpha} + 2m_{2\alpha}) = 2n + 1.$$

We have the Iwasawa decomposition  $G = KAN = KNA$  and the corresponding integral formulas:

$$(1.3) \quad \int_G f(g) dg = \int_K \int_{-\infty}^{+\infty} \int_N f(ka_t n) e^{2\rho t} dk dt dn$$

$$(1.4) \quad = \int_K \int_N \int_{-\infty}^{+\infty} f(kna_t) dk dn dt$$

for  $f \in C_c(G)$ . We adopt the usual notation  $C_c(G)$  for the space of continuous functions on  $G$  with compact support. In the above formulas,  $dn = dw dz$  ( $n = n(w, z)$ ) and  $dk$  is the normalized Haar measure on  $K$ .

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \quad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every  $g \in G$  can be written as  $g = k_1 k_2 a_t k'_2$  for uniquely determined  $k_1 \in K_1$ ,  $t \geq 0$  and for some  $k_2, k'_2 \in K_2$ . Writing  $g = [g_{ij}]_{i,j=0}^n$ , we have

$$(1.5) \quad k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|.$$

If  $g \notin K$ , then  $t > 0$  and  $k_2, k'_2$  are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After  $dg$  is normalized according to (1.3), the corresponding integral formula is

$$(1.6) \quad \int_G f(g) dg = \frac{1}{2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1} \int_{K_2} \int_0^\infty \int_{K_2} f(k_1 k_2 a_t k'_2) \Delta(t) dk_1 dk_2 dt dk'_2$$

where

$$(1.7) \quad \Delta(t) := 2^{2\rho} (\sinh t)^{4n-1} (\cosh t)^3.$$

## 2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G; \chi_l)$

Let  $\mathbf{N}/2$  be the set of nonnegative half-integers  $\{0, 1/2, 1, 3/2, \dots\}$ . Since  $K_1 \cong \mathrm{Sp}(1)$  is isomorphic to  $\mathrm{SU}(2)$ ,  $\mathbf{N}/2$  parametrizes the set of equivalence classes of unitary irreducible representations of  $K_1$ . We denote with the same symbol  $\tau_l$  either the equivalence class corresponding to the parameter  $l$  or a fixed representative for it. Thus  $\tau_l$  is a unitary irreducible representation of  $K_1$  in a Hilbert space  $V_l$  of dimension  $d_l = 2l + 1$ . We extend  $\tau_l$  to a representation of  $K$  by setting  $\tau_l \equiv \mathbf{1}$  on  $K_2$ . Each  $\tau_l$  is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character  $\chi_l = \mathrm{tr} \tau_l$  of  $\tau_l$  satisfies  $\chi_l(k^{-1}) = \chi_l(k)$ ,  $k \in K$ .

We denote by  $\mathcal{D}(G; \tau_l)$  the convolution algebra of the compactly supported  $C^\infty$  maps  $F: G \rightarrow \mathrm{End}(V_l)$  satisfying