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Note that  $\Phi$  is a regular morphism defined on  $U$  and  $\Theta$  is a regular morphism defined on  $V$ . Since

$$\Phi \Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta \Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps  $\Phi$  and  $\Theta$  induce regular morphisms  $\Phi: U \cap \Phi^{-1}(V) \rightarrow V \cap \Theta^{-1}(U)$  and  $\Theta: V \cap \Theta^{-1}(U) \rightarrow U \cap \Phi^{-1}(V)$ . To show that  $\Phi$  and  $\Theta$  are birational inverses to each other, it is enough to check that  $U \cap \Phi^{-1}(V)$  and  $V \cap \Theta^{-1}(U)$  are non-empty.

Note that  $(D_1, D_2) \in U \cap \Phi^{-1}(V)$  if and only if  $(D_1, D_2) \in U$  and

$$l_m(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - m) = 0.$$

Since  $m(D_1, D_2) \sim_m D_1 + D_2 - \pi P_0$ , the above equations are equivalent to

$$l_m(D_2) = 1, \quad l(D_2 - m) = 0.$$

Applying Lemma 3.3 to the divisor  $D_0 = 0$ , we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_m(D) = 0, \quad l(D - m) = 0\}$$

is open and non-empty. Since  $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$  is irreducible, the set  $U \cap ((X - S)^{(\pi)} \times V_0)$  is also open and non-empty. This set is exactly  $U \cap \Phi^{-1}(V)$ . So  $U \cap \Phi^{-1}(V)$  is non-empty.

Similarly  $V \cap \Theta^{-1}(U)$  is also non-empty. This completes the proof of the proposition.

#### 4. FROM BIRATIONAL GROUPS TO ALGEBRAIC GROUPS

Let  $k$  be an algebraically closed field, let  $V$  be a connected nonsingular variety over  $k$ , and let  $m: V \times V \rightarrow V$ ,  $(a, b) \mapsto ab$  be a rational map satisfying  $(ab)c = a(bc)$ . Assume the rational maps  $\Phi(a, b) = (a, ab)$  and  $\Psi(a, b) = (b, ab)$  are birational. Then there exist open subsets  $X_\Phi$ ,  $Y_\Phi$ ,  $X_\Psi$  and  $Y_\Psi$  in  $V \times V$  such that  $\Phi$  induces an isomorphism  $X_\Phi \cong Y_\Phi$  and  $\Psi$  induces an isomorphism  $X_\Psi \cong Y_\Psi$ . Put  $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$ .

It is convenient to write the formulae for  $\Phi^{-1}$  and  $\Psi^{-1}$  as  $\Phi^{-1}(a, b) = (a, a^{-1}b)$  and  $\Psi^{-1}(a, b) = (ba^{-1}, a)$ .

LEMMA 4.1. *Replacing  $V$  by an open subset, we may assume the two projections  $p_i: Z \rightarrow V$  ( $i = 1, 2$ ) are surjective.*

*Proof.* Note that the two projections  $p_i: V \times V \rightarrow V$ , ( $i = 1, 2$ ) are flat since  $V \rightarrow \text{spec}(k)$  is flat. So the  $p_i$  are open by [EGA] IV, §2.4.6. Hence the  $p_i(Z)$  are open. Let  $V' = p_1(Z) \cap p_2(Z)$ . We claim  $V'$  has the property stated in the lemma. Let  $C = V - V'$  and let  $A = (C \times V) \cup (V \times C)$ . The subset  $X_\Phi'$  of  $V' \times V'$  corresponding to  $X_\Phi$  is the complement in  $X_\Phi$  of  $S = (X_\Phi \cap A) \cup \Phi^{-1}(Y_\Phi \cap A)$ . We claim that if the fiber of  $p_1: X_\Phi \rightarrow V$  at  $v \in V$  is contained in  $S$ , then  $v \in C$ . Thus  $p_1: X_\Phi' \rightarrow V'$  is surjective.

Let us prove the claim. Assume  $(v \times V) \cap X_\Phi \subset S$ , but  $v \notin C$ . We have  $(v \times V) \cap X_\Phi \subset S \subset A \cup \Phi^{-1}(A) \subset (C \times V) \cup (V \times C) \cup \Phi^{-1}(C \times V) \cup \Phi^{-1}(V \times C)$ .

Since  $V$  is irreducible, we must have

$$(v \times V) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V), \quad \text{or} \quad \Phi^{-1}(V \times C).$$

Since  $v \notin C$ , we have

$$(v \times V) \cap X_\Phi \not\subset C \times V, \quad \Phi^{-1}(C \times V).$$

So

$$(v \times V) \cap X_\Phi \subset V \times C \quad \text{or} \quad \Phi^{-1}(V \times C).$$

Assume  $(v \times V) \cap X_\Phi \subset V \times C$ . Note that since  $v \notin C$ , we have  $v \in V'$ . Hence  $(v \times V) \cap X_\Phi$  is not empty. So we have

$$\begin{aligned} \dim V &= \dim((v \times V) \cap X_\Phi) = \dim(((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\leq \dim(v \times C) < \dim V, \end{aligned}$$

that is,  $\dim V < \dim V$ . This is impossible.

Assume  $(v \times V) \cap X_\Phi \subset \Phi^{-1}(V \times C)$ . Then  $\Phi((v \times V) \cap X_\Phi) \subset V \times C$ . Since  $\Phi$  is birational, we have

$$\begin{aligned} \dim V &= \dim \Phi((v \times V) \cap X_\Phi) = \dim(\Phi((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\leq \dim(v \times C) < \dim V, \end{aligned}$$

which is again impossible. So we must have  $v \in C$ .

Next we show that if the fiber of  $p_2: X_\Phi \rightarrow V$  at  $v \in V$  is contained in  $S$ , then  $v \in C$ , and hence  $p_2: X_\Phi' \rightarrow V'$  is surjective.

Assume  $(V \times v) \cap X_\Phi \subset S$  but  $v \notin C$ . As before we have

$$(V \times v) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V) \quad \text{or} \quad \Phi^{-1}(V \times C).$$

Since  $v \notin C$ , we have  $(V \times v) \cap X_\Phi \not\subset V \times C$ . By counting dimensions, one can show  $(V \times v) \cap X_\Phi \not\subset C \times V$ . Since  $\Phi^{-1}(C \times V) \subset C \times V$ , we have  $(V \times v) \cap X_\Phi \not\subset \Phi^{-1}(C \times V)$ . So we can only have  $(V \times v) \cap X_\Phi \subset \Phi^{-1}(V \times C)$ . Then we have a rational map

$$V \xrightarrow{\iota_1} (V \times v) \cap X_\Phi \xrightarrow{\Phi} V \times C \xrightarrow{p_2} C,$$

where  $\iota_1(x) = (x, v)$ . This map  $p_2 \Phi \iota_1: V \rightarrow C$  is nothing but  $x \mapsto xv$  and it is birational. (Its birational inverse is  $p_1 \Psi^{-1} \iota_2$ , where  $\iota_2(x) = (v, x)$ .) So  $V$  is birational to  $C$ . This is impossible since  $\dim V \neq \dim C$ . So we must have  $v \in C$ . This finishes the proof of the surjectivity of  $p_2: X_\Phi' \rightarrow V'$ .

Similarly  $p_i: X_\Phi', Y_\Phi', X_\Psi', Y_\Psi' \rightarrow V'$  are surjective. Since the fibers of  $p_i: V \times V \rightarrow V$  are irreducible, the projection  $p_i: Z' = X_\Phi' \cap Y_\Phi' \cap X_\Psi' \cap Y_\Psi' \rightarrow V'$  is also surjective.

Having replaced  $V$  as in Lemma 4.1, we may assume  $V$  satisfies the following properties:

**PROPERTY 4.2.** *There exists an open set  $Z \subset V \times V$  such that  $\Phi, \Phi^{-1}, \Psi$ , and  $\Psi^{-1}$  are defined on  $Z$ , the restrictions  $\Phi|_Z$  and  $\Psi|_Z$  are open immersions, and the projections  $p_i: Z \rightarrow V$  are surjective. Hence for every  $v \in V$ , the maps  $\Phi, \Phi^{-1}, \Psi$  and  $\Psi^{-1}$  are defined at  $(v, x)$  and at  $(x, v)$ , provided  $x$  is generic, i.e. lies in an open set.*

**LEMMA 4.3.** *Assume 4.2 holds. Denote the closure of the graph of  $m$  in  $V \times V \times V$  by  $\Gamma$ . Then the projections  $p_{ij}: \Gamma \rightarrow V \times V$  ( $1 \leq i < j \leq 3$ ) are open immersions.*

*Proof.* By [EGA] III, §4.4.9, it suffices to show that the maps  $p_{ij}$  are set-theoretically injective. Let  $x$  be a point of  $V$ . The two rational maps  $\Gamma \rightarrow V$  defined by

$$(a, b, c) \mapsto (xa)b \quad \text{and} \quad (a, b, c) \mapsto xc$$

are equal by the associative law. Let  $(a, b, c), (a, b, c') \in \Gamma$ . Choose  $x$  so that  $(xa)b$  is defined and  $(x, c), (x, c') \in Z$ . Then  $xc = (xa)b = xc'$ . Hence  $\Phi(x, c) = \Phi(x, c')$ . Since  $\Phi$  is an open immersion on  $Z$ , we have  $(x, c) = (x, c')$ . Hence  $c = c'$ . This shows that  $p_{12}: \Gamma \rightarrow V \times V$  is injective. Similarly one can show the other projections are injective.

We will now expand  $V$  to the group we want by glueing translates of  $V$ . Let  $s$  be a point of  $V$  and let  $V_s$  be a copy of  $V$  thought of as the

translate  $V_s = \{vs \mid v \in V\}$ . The subset  $W_s = (V \times s \times V) \cap \Gamma$  is closed in  $V \times s \times V \cong V \times V$ , and the two projections  $W_s \rightarrow V$  are open immersions because they are the base extensions of the open immersions  $p_{ij}: \Gamma \rightarrow V \times V$  by the base changes  $V \times s \rightarrow V \times V$  and  $s \times V \rightarrow V \times V$ , respectively. Therefore  $W_s$  defines glueing data and yields a separated scheme  $V' = V \cup_{W_s} V_s$ .

LEMMA 4.4.  *$V$  is an open dense subset of  $V'$  and  $V'$  satisfies 4.2.*

*Proof.* Since  $xs$  is defined for generic  $x \in V$ , the set  $V \cap V_s$  is not empty. So  $V'$  is irreducible and  $V$  is dense in  $V'$ . We have

$$V' \times V' = (V \times V) \cup (V \times V_s) \cup (V_s \times V) \cup (V_s \times V_s).$$

For every point  $v \in V$ , denote by  $v_s$  the point  $v$  considered as a point in  $V_s$ . Note that if  $(v, s) \in Z$ , then  $vs \in V$  and  $v_s \in V_s$  are glued together in  $V'$ . Define  $R_s: V \rightarrow V_s$  by  $v \mapsto v_s$ . Let

$$W_1 = \{(a, b) \in V \times V \mid (a, b), (s, a) \text{ and } (b, sa^{-1}) \text{ are all in } Z\}.$$

This is a non-empty open subset of  $Z$ . Take  $Z_1 = (\text{id} \times R_s)(W_1) \subset V \times V_s$ . We define  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$  and  $\Psi^{-1}$  on  $Z_1$  by

$$\begin{aligned} \Phi(a, b_s) &= (a, (ab)_s) \in V \times V_s, \\ \Psi(a, b_s) &= (b_s, (ab)_s) \in V_s \times V_s, \\ \Phi^{-1}(a, b_s) &= (a, (a^{-1}b)_s) \in V \times V_s, \\ \Psi^{-1}(a, b_s) &= (b(sa^{-1}), a) \in V \times V \end{aligned}$$

for any  $(a, b_s) \in Z_1$ . Let

$$W_2 = \{(a, b) \in V \times V \mid (a, b), (s, b), (a, sb), (s, a^{-1}b) \text{ and } (bs^{-1}, a) \text{ are all in } Z\}.$$

This is a non-empty open subset of  $Z$ . Take  $Z_2 = (R_s \times \text{id})(W_2) \subset V_s \times V$ . We define  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$ , and  $\Psi^{-1}$  on  $Z_2$  by

$$\begin{aligned} \Phi(a_s, b) &= (a_s, a(sb)) \in V_s \times V, \\ \Psi(a_s, b) &= (b, a(sb)) \in V \times V, \\ \Phi^{-1}(a_s, b) &= (a_s, s^{-1}(a^{-1}b)) \in V_s \times V, \\ \Psi^{-1}(a_s, b) &= ((bs^{-1})a^{-1}, a_s) \in V \times V_s \end{aligned}$$

for any  $(a_s, b) \in Z_2$ .

Let  $Z' = Z \cup Z_1 \cup Z_2$ . It is an open subset of  $V' \times V'$ , and  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$ ,  $\Psi^{-1}$  are defined on it. One can show that  $\Phi|_{Z'}$  and  $\Psi|_{Z'}$  are open

immersions. Given  $v \in V'$ , we need to show there exists  $x \in V'$  such that  $(x, v)$  and  $(v, x)$  are in  $Z'$ . This is true if  $v \in V$  by the property of  $Z$ . If  $v \in V_s$ , then  $v = a_s$  for some  $a \in V$ . We leave it to the reader to show that  $(x, a_s) \in Z_1$  and  $(a_s, x) \in Z_2$  for generic  $x$  in  $V$ . This completes the proof of the lemma.

The above lemma allows us to replace  $V$  by  $V'$ , hence to expand  $V$  whenever there exists a point  $s$  in  $V$  such that  $vs$  is not defined for all  $v \in V$ , and we can expand  $V'$  if there exists a point  $s' \in V'$  such that  $v's'$  is not defined for all  $v' \in V'$ . Denote the result of finitely many such expansions also by  $V'$ , and let  $U \subset V \times V \times V'$  be the closure of  $\Gamma$ . By Lemma 4.3 applied to  $V'$ , the projection  $p_{12}: U \rightarrow V \times V$  is an open immersion. Its image is the set of points  $(a, b)$  such that  $m: V \times V \rightarrow V'$  is defined at  $(a, b)$ . If  $V \times s \not\subset p_{12}(U)$  for some point  $s$  in  $V$ , then replacing  $V'$  by  $V' \cup V_s'$  increases both  $V'$  and  $p_{12}(U)$ . Using noetherian induction on open subschemes of  $V \times V$ , we may assume that after finitely many expansions,  $V \times s \subset p_{12}(U)$  for all points  $s \in V$ . Then we have  $p_{12}(U) = V \times V$ .

**PROPOSITION 4.5.** *Let  $V$ ,  $V'$ , and  $U$  be as above. If  $p_{12}(U) = V \times V$ , then the operation  $m: V' \times V' \rightarrow V'$  is everywhere defined on  $V'$  and makes  $V'$  an algebraic group.*

*Proof.* Take  $(a', b')$  in  $V' \times V'$ . Choose a point  $x$  so that  $a'x$  and  $x^{-1}b'$  are both defined and lie in  $V$ . Then we can define  $m(a', b') = (a'x)(x^{-1}b')$ . Similarly one can define  $a'^{-1}b'$  and  $b'a'^{-1}$ . In this way we extend  $m$ ,  $\Phi$ ,  $\Psi$ ,  $\Phi^{-1}$  and  $\Psi^{-1}$  to  $V' \times V'$ . The verification of the group axioms is routine and is omitted.

## 5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on  $(X - S)^{(\pi)}$ . The algebraic group associated to this birational group is called the *generalized jacobian* of  $X_m$  and is denoted by  $J_m$ . It is a commutative algebraic group.

Let  $D_0$  be a divisor on  $X$  prime to  $S$  of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{D \in (X - S)^{(\pi)} \mid l_m(D + D_0) = 1, \quad l(D + D_0 - m) = 0\}$$

is a non-empty open subset of  $(X - S)^{(\pi)}$ . We have the following