is an algebra homomorphism of $\mathcal{D}(G;\chi_l)$ into $\mathbf{C}$.

Set

$$\alpha_{l,s}(ka_tn) = \frac{1}{d_l}\chi_l(k)e^{-(s+\rho)t}\,. \tag{3.21}$$

Since $f = f * d_l\chi_l$ and $\chi_l(k^{-1}) = \chi_l(k)$ for $k\in K$, for every $f\in\mathcal{D}(G;\chi_l)$

$$\begin{aligned}\lambda_s(f) &= \frac{1}{d_l}\int_K\int_{-\infty}^{\infty}\int_N f(ka_tn)\chi_l(k)e^{(-s+\rho)t}\,dk\,dt\,dn\\ &= \int_G f(g)\alpha_{l,s}(g)\,dg\\ &= \int_G f(g)\int_K \alpha_{l,s}(kgk^{-1})\,dk\,dg\\ &= \int_G f(g)\zeta_{l,s}(g)\,dg \end{aligned}\tag{3.22}$$

with

$$\zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1})\,dk\,. \tag{3.23}$$

One easily checks that $\zeta_{l,s}$ satisfies $\zeta_{l,s} = \zeta_{l,s}{}^0$, $\zeta_{l,s}*d_l\chi_l = \zeta_{l,s}$ and $\zeta_{l,s}(e) = 1$. Thus $\zeta_{l,s}$ is a τ_l-spherical function. It will be shown in the next section that any τ_l-spherical function is of the form (3.24).

By Remark 2.3, we have

$$\zeta_{l,s}(g) = \frac{1}{d_l}\chi_l(k_1)\zeta_{l,s}(a_t) \quad \text{for } g = k_1k_2a_tk_2'\,, \tag{3.24}$$

so $\zeta_{l,s}$ is uniquely determined by its restriction to A.

4. THE DIFFERENTIAL EQUATION FOR THE τ_l-SPHERICAL FUNCTIONS

For a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$, let $\mathfrak{u}_{\mathbf{C}}$ denote the complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by $\mathfrak{u}$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{u})$ of $\mathfrak{u}_{\mathbf{C}}$ is considered as a subalgebra of $\mathfrak{U}(\mathfrak{g})$.

The representation τ_l of K_1 induces differentiated representations of the Lie algebra $\mathfrak{k}_1$ of K_1 and of the universal enveloping algebra $\mathfrak{U}(\mathfrak{k}_1)$ of $(\mathfrak{k}_1)_{\mathbf{C}}$. We indicate these representations with the same letter τ_l. Let $\mathfrak{k}_2$ be the Lie algebra of K_2. Every element $Y\in\mathfrak{k}_{\mathbf{C}}$ can be uniquely decomposed as

$Y = Y^{(1)} + Y^{(2)}$ with $Y^{(j)} \in (\mathfrak{k}_j)_{\mathbf{C}}$, $j = 1, 2$. The symbol χ_l will also be used for the $\mathbf{C}$-linear map on $\mathfrak{U}(\mathfrak{k})$ defined by

$$\chi_l(Y_1 \cdots Y_m) := \operatorname{tr}\left[\tau_l(Y_1^{(1)}) \cdots \tau_l(Y_m^{(1)})\right]$$

for $Y_1, \ldots, Y_m \in \mathfrak{k}_{\mathbf{C}}$.

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ gives $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{k})\mathfrak{U}(\mathfrak{a})\mathfrak{U}(\mathfrak{n}) \cong \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a}) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbf{C}}$. Let $P\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a})$ be the corresponding projection. For $s \in \mathbf{C}$, let e_s be the $\mathbf{C}$-linear map on $\mathfrak{U}(\mathfrak{a})$ defined by

$$e_s(L^m) := (-1)^m (s+\rho)^m \qquad \text{for every integer } m \geq 0\,.$$

Define $p_{l,s}\colon \mathfrak{U}(\mathfrak{g}) \to \mathbf{C}$ to be the composition $p_{l,s} := \left(\frac{1}{d_l}\chi_l \otimes e_s\right) \circ P$, where as before $d_l = \dim \tau_l$.

4.1. PROPOSITION. *Let $\zeta_{l,s}$ be the function defined by Formula (3.23). For every $D \in \mathfrak{U}(\mathfrak{g})^K$ and $g \in G$*

$$\zeta_{l,s}(g; D) = p_{l,s}(D)\zeta_{l,s}(g)\,. \tag{4.25}$$

Proof. Because of Theorem 3.1, $\zeta_{l,s}$ is an eigenfunction of every $D \in \mathfrak{U}(\mathfrak{g})^K$. The eigenvalue corresponding to $D \in \mathfrak{U}(\mathfrak{g})^K$ is $\zeta_{l,s}(e; D)$ because $\zeta_{l,s}(e) = 1$. Since D is K-invariant, $\zeta_{l,s}(e; D) = \alpha_{l,s}(e; D)$. Write $D = \sum_i y_i x_i + \sum_j n_j$ with $y_i \in \mathfrak{U}(\mathfrak{k})$, $x_i \in \mathfrak{U}(\mathfrak{a})$ and $n_j \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbf{C}}$. Then $\alpha_{l,s}(e; D) = \sum_i \alpha_{l,s}(e; y_i x_i)$ because $\alpha_{l,s}(gn) = \alpha_{l,s}(g)$ for $g \in G$ and $n \in N$. To compute $\alpha_{l,s}(e; y_i x_i)$, assume without loss of generality that $x_i = L^{m_i}$ and that $y_i = Y_1 \cdots Y_m$ with $Y_j \in \mathfrak{k}$. The definition of $\alpha_{l,s}$ gives

$$\alpha_{l,s}(e; y_i x_i) = \frac{1}{d_l}\chi_l(y_i)(-1)^m (s+\rho)^m = p_{l,s}(y_i x_i)\,.$$

Thus $\zeta_{l,s}(e; D) = p_{l,s}(D)$. □

Let $\delta_l(D)$ denote the τ_l-radial component on $A^+ := \{a_t : t > 0\}$ of the differential operator $D \in \mathfrak{U}(\mathfrak{g})$; that is, the unique differential operator on A^+ satisfying

$$f(a_t; \delta_l(D)) = f(a_t; D)$$

for all $f \in \mathcal{D}(G; \chi_l)$ and $t > 0$. Proposition 4.1 immediately implies

4.2. COROLLARY. *$\zeta_{l,s}$ is an eigenfunction of the τ_l-radial component on A^+ of every K-invariant differential operator on G:*

$$\zeta_{l,s}(a_t; \delta_l(D)) = p_{l,s}(D)\zeta_{l,s}(a_t) \qquad (D \in \mathfrak{U}(\mathfrak{g})^K,\ t > 0)\,. \tag{4.26}$$

We now write (4.26) explicitly in the case D is the Casimir operator ω of $\mathfrak{g}$. Let B denote the Cartan-Killing form of $\mathfrak{g}_{\mathbf{C}}$ ($\cong \mathfrak{sp}(1+n,\mathbf{C})$). If $X, Y \in \mathfrak{sp}(1,n)$, then

$$B(X,Y) = 4(n+2)\,\Re\,\mathrm{tr}(XY)$$

where $\Re$ denotes the quaternionic real part: $\Re q = \frac{q+\bar{q}}{2}$ for $q \in \mathbf{H}$. The bilinear form $B_\theta(X,Y) := -B(X,\theta Y)$ is an inner product on $\mathfrak{g}$. Orthonormality will be considered with respect to B_θ.

Let $\{Z_j\}_{j=1}^m$ ($m := 2n^2+n$) and $\{X_{\beta,j}\}_{j=1}^{m_\beta}$ ($\beta \in \{\alpha, 2\alpha\}$) be orthonormal bases in $\mathfrak{m}$ and in $\mathfrak{g}_\beta$ respectively. Define $X_{-\beta,j} = -\theta(X_{\beta,j})$ for $\beta \in \{\alpha,2\alpha\}$ and $j = 1,\dots,m_\beta$. Then $\{X_{-\beta,j}\}_{j=1}^{m_\beta}$ is an orthonormal basis for $\mathfrak{g}_{-\beta}$, and $B(X_{\beta,i}, X_{-\beta,j}) = \delta_{ij}$. Moreover, for all $j = 1,\dots,m_\beta$, $H_\beta := [X_{\beta,j}, X_{-\beta,j}]$ is the unique element of $\mathfrak{a}$ satisfying $B(H_\beta, L) = \beta(L)$, i.e.

$$H_\beta = \frac{h_\beta}{8(n+2)}\,L \quad \text{with} \quad h_\beta = \begin{cases} 1 & \text{if } \beta = \alpha \\ 2 & \text{if } \beta = 2\alpha\,. \end{cases}$$

Set $H_1 := \frac{L}{\sqrt{8(n+2)}}$, a unit vector in $\mathfrak{a}$. Then, if $D_{\beta,j} := X_{\beta,j}X_{-\beta,j} + X_{-\beta,j}X_{\beta,j}$, we have (cf. [GaV], p. 132)

$$\begin{aligned} \omega &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} \sum_{j=1}^{m_\beta} D_{\beta,j} \\ &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} m_\beta H_\beta + 2 \sum_{\beta\in\{\alpha,2\alpha\}} \sum_{j=1}^{m_\beta} X_{\beta,j}X_{-\beta,j} \end{aligned} \tag{4.27}$$

where

$$\omega_{\mathfrak{m}} := -\sum_{j=1}^{m} Z_j^2\,. \tag{4.28}$$

Hence

$$P(\omega) = \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta\in\{\alpha,2\alpha\}} m_\beta H_\beta = \omega_{\mathfrak{m}} + \frac{L^2 + 2\rho L}{B(L,L)}\,,$$

from which we conclude

$$p_{l,s}(\omega) = p_{l,s}(\omega_{\mathfrak{m}}) + \frac{(s+\rho)^2 - 2\rho(s+\rho)}{B(L,L)} = \frac{1}{d_l}\chi_l(\omega_{\mathfrak{m}}) + \frac{s^2-\rho^2}{8(n+2)}\,. \tag{4.29}$$

To compute $\delta_l(\omega)$ we use Formula (4.27). Observe first that if $f \in \mathcal{D}(G;\chi_l)$ and $Y \in \mathfrak{U}(\mathfrak{k})$, then $f(a_t;Y) = \frac{1}{d_l}\chi_l(Y)$. Hence $\delta_l(Y) = \frac{1}{d_l}\chi_l(Y)$. In particular,

$$\delta_l(\omega_{\mathfrak{m}}) = \frac{1}{d_l}\chi_l(\omega_{\mathfrak{m}}) . \tag{4.30}$$

Write

$$X_{\beta,j} = Y_{\beta,j} + P_{\beta,j} \qquad \text{with} \qquad Y_{\beta,j} \in \mathfrak{k}, P_{\beta,j} \in \mathfrak{p} . \tag{4.31}$$

A standard computation (cf. e.g. [W2], p. 278) then gives for $f \in \mathcal{D}(G;\chi_l)$ and $t > 0$

$$f(a_t;D_{\beta,j}) = \coth(t\beta(L))f(a_t;H_\beta) + \frac{4}{d_l}\frac{1-\cosh(t\beta(L))}{\sinh^2(t\beta(L))}\chi_l({Y_{\beta,j}}^2)f(a_t)$$

i.e.

$$\delta_l(D_{\beta,j}) = \coth(t\beta(L))H_\beta + \frac{4}{d_l}\frac{1-\cosh(t\beta(L))}{\sinh^2(t\beta(L))}\chi_l({Y_{\beta,j}}^2) . \tag{4.32}$$

Notice that $\chi_l(Y^2_{\alpha,j}) = 0$ for all $j = 1, \ldots, m_\alpha$.

For $h = i,j,k$, let Y_h denote the tangent vector at e to the 1-parameter subgroup $t \mapsto \cos t + h\sin t$ in $\mathrm{Sp}(1)$. Explicit choices of the orthonormal bases in $\mathfrak{m}$ and $\mathfrak{g}_{2\alpha}$ prove that

$$\chi_l(\omega_{\mathfrak{m}}) = -2\sum_{j=1}^{3}\chi_l(Y^2_{2\alpha,j}) = -\frac{1}{8(n+2)}\sum_{h\in\{i,j,k\}}\mathrm{tr}\left[\tau_l(Y_h)^2\right] . \tag{4.33}$$

As shown in [T1], p. 381, there exists an orthonormal basis $\{v_p\}_{p=-l}^{l}$ in V_l such that

$$\begin{aligned}
\tau_l(Y_i)v_p &= -2ipv_p \\
\tau_l(Y_j)v_p &= -i\alpha^l_{p+1}v_{p+1} \\
\tau_l(Y_k)v_p &= -\alpha^l_{p+1}v_{p+1} + \alpha^l_p v_{p-1}
\end{aligned}$$

where

$$\alpha^l_p := [(l+p)(l-p+1)]^{1/2} .$$

It follows that for $h = i,j,k$

$$\mathrm{tr}\left[\tau_l(Y_h)^2\right] = -\frac{4}{3}l(l+1)(2l+1) . \tag{4.34}$$

Identify A with $\mathbf{R}$ and L with $\frac{d}{dt}$ under the isomorphism $t \mapsto \exp(tL) = a_t$. Formulas (4.27), (4.30) and (4.32)–(4.34) then prove the following proposition.

4.3. PROPOSITION. *Let τ_l be an irreducible unitary representation of K_1 of dimension $2l+1$. Then*

1. *The τ_l-radial component of the Casimir operator ω is*

$$\delta_l(\omega)=\frac{1}{8(n+2)}\left\{\frac{d^2}{dt^2}+[(4n-1)\coth t+3\tanh t]\frac{d}{dt}+\frac{4l(l+1)}{\cosh^2 t}+4l(l+1)\right\}.$$

2. *For every $s\in\mathbf{C}$*

$$p_{l,s}(\omega)=\frac{1}{8(n+2)}[4l(l+1)+s^2-\rho^2]. \tag{4.35}$$

3. *For every $s\in\mathbf{C}$, the function $\zeta_{l,s}(t):=\zeta_{l,s}(a_t)$ satisfies the differential equation on $(0,+\infty)$*

$$u''+[(4n-1)\coth t+3\tanh t]u'+\frac{4l(l+1)}{\cosh^2 t}u=(s^2-\rho^2)\,u. \tag{4.36}$$

The substitution $v(t)=(\cosh t)^{-2l}u(t)$ transforms the differential equation (4.36) into the Jacobi differential equation (cf. [K2], p. 6)

$$v''+[(4n-1)\coth t+(4l+3)\tanh t]v'=(s^2-\widetilde{\rho}^2)v \tag{4.37}$$

with parameters $\alpha=2n-1$ and $\beta=2l+1$. Here $\widetilde{\rho}:=\alpha+\beta+1=\rho+2l$. The Jacobi function

$$\begin{aligned}\phi_{is}^{(2n-1,2l+1)}(t):=&F\left(\frac{\tilde{\rho}+s}{2},\frac{\tilde{\rho}-s}{2};2n;-\sinh^2 t\right)\\=&F\left(\frac{\rho+s}{2}+l,\frac{\rho-s}{2}+l;2n;-\sinh^2 t\right)\end{aligned} \tag{4.38}$$

is the unique solution v to (4.37) satisfying $v(0)=1$, $v'(0)=0$. (In (4.38), $F(a,b;c;z)$ denotes the analytic branch on $\mathbf{C}\setminus[1,\infty)$ of the hypergeometric function.)

The τ_l-spherical function $\zeta_{l,s}$ is therefore explicitly given by

$$\begin{aligned}\zeta_{l,s}(t):=\zeta_{l,s}(a_t)&=(\cosh t)^{2l}\,\phi_{is}^{(2n-1,2l+1)}(t)\\&=(\cosh t)^{2l}F\left(\frac{\rho+s}{2}+l,\frac{\rho-s}{2}+l;2n;-\sinh^2 t\right).\end{aligned} \tag{4.39}$$

Formula (4.39) has been previously determined by Takahaski ([T2], Formula (7), p. 225) by direct integration of (3.23), using the following expression of χ_l in terms of Gegenbauer polynomials:

$$\chi_l(k_1)=C^1_{2l}(\Re k_1)=\frac{\sin((2l+1)\vartheta)}{\sin\vartheta}\qquad\text{if}\quad\Re k_1=\cos\vartheta. \tag{4.40}$$

Formula (4.35) shows that $p_{l,s}(\omega)$ is an even function of s which assumes arbitrary complex values as s varies in $\mathbf{C}$. The following corollary can therefore be deduced from Theorem 3.1 and Proposition 4.3.

4.4. COROLLARY. *The τ_l-spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p. 227)

$$\zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t,\tau,u)\zeta_{l,s}(u)\Delta(u)\,du \tag{4.41}$$

where Δ is as in (1.7) and the kernel $K_l(t,\tau,u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2\cosh t \cosh \tau \cosh u}\,.$$

Then

$$K_l(t,\tau,u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\,\Gamma(2n-\frac{1}{2})}\,\frac{(\cosh t\cosh\tau\cosh u)^{2n-3}}{(\sinh t\sinh\tau\sinh u)^{4n-2}}\,(1-B^2)^{2n-\frac{3}{2}} \times F\Big(2n+2l, 2n-2l-2; 2n-\frac{1}{2}; \frac{1}{2}(1-B)\Big) \tag{4.42}$$

if $B < 1$, and $K_l(t,\tau,u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n-1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l-SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G\int_G \zeta(x^{-1}y)f(x)\overline{f(y)}\,dx\,dy \geq 0\,.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) are the Lie algebras of G and K, respectively. A (strongly continuous) representation T of G on a Banach space $\mathcal{H}$ is denoted by $(T, \mathcal{H})$. We may simply speak of the representation T if $\mathcal{H}$ is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let $\widehat{K}$ denote the set of equivalence classes