

# 5. The positive definite $\tau_I$ -spherical functions

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4.4. COROLLARY. *The  $\tau_l$ -spherical functions are exactly the functions  $\{\zeta_{l,s} : s \in \mathbf{C}\}$  given by Formulas (3.24) and (4.39). Further,  $\zeta_{l,s}$  satisfies  $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$  for all  $g \in G$ . Moreover,  $\zeta_{l,s} = \zeta_{l,s'}$  if and only if  $s = \pm s'$ .*

The functional equation (3.15) with  $g_1 = a_t$  and  $g_2 = a_\tau$  becomes (cf. [T2], Théorème 1, p.227)

$$(4.41) \quad \zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t, \tau, u)\zeta_{l,s}(u)\Delta(u) du$$

where  $\Delta$  is as in (1.7) and the kernel  $K_l(t, \tau, u)$  is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2 \cosh t \cosh \tau \cosh u}.$$

Then

$$(4.42) \quad K_l(t, \tau, u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n - \frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1 - B^2)^{2n-\frac{3}{2}} \\ \times F\left(2n + 2l, 2n - 2l - 2; 2n - \frac{1}{2}; \frac{1}{2}(1 - B)\right)$$

if  $B < 1$ , and  $K_l(t, \tau, u) := 0$  if  $B \geq 1$ . Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all  $l \in \mathbf{R}$  satisfying  $2n - 1 > 2l \geq 0$ .

## 5. THE POSITIVE DEFINITE $\tau_l$ -SPHERICAL FUNCTIONS

A continuous function  $\zeta$  on a locally compact group  $G$  is said to be *positive definite* if for every  $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y)f(x)\overline{f(y)} dx dy \geq 0.$$

In this section we establish which among the  $\zeta_{l,s}$  are positive definite.

Let us first introduce some notation and recall some definitions. Let  $G$  be a semisimple Lie group with finite center, and let  $K$  be a maximal compact subgroup of  $G$ .  $\mathfrak{g}$  and  $\mathfrak{k}$  ( $\subset \mathfrak{g}$ ) are the Lie algebras of  $G$  and  $K$ , respectively. A (strongly continuous) representation  $T$  of  $G$  on a Banach space  $\mathcal{H}$  is denoted by  $(T, \mathcal{H})$ . We may simply speak of the representation  $T$  if  $\mathcal{H}$  is understood. Irreducibility for  $T$  always means topological irreducibility (= no closed proper invariant subspaces). Let  $\widehat{K}$  denote the set of equivalence classes

of finite dimensional irreducible representations of  $K$ . We say that  $\tau \in \widehat{K}$  occurs in  $T|_K$  if there exists a finite dimensional  $T|_K$ -invariant subspace  $V$  of  $\mathcal{H}$  so that  $(T|_K, V) \in \tau$ . The linear span of all these subspaces  $V$  is the  $K$ -isotypic subspace of  $\mathcal{H}$  of type  $\tau$ , denoted  $\mathcal{H}(\tau)$ . If  $d_\tau$  is the dimension of  $\tau$  and  $\chi_\tau$  is its character, then

$$E_T(\tau) = d_\tau \int_K T(k^{-1})\chi_\tau(k) dk$$

is a continuous projection of  $\mathcal{H}$  onto  $\mathcal{H}(\tau)$ . We set  $\mathcal{H}_K = \sum_{\tau \in \widehat{K}} \mathcal{H}(\tau)$ .  $T$  is said to be  $K$ -finite if  $\dim \mathcal{H}(\tau) < \infty$  for all  $\tau \in \widehat{K}$ . A Hilbert representation  $(T, \mathcal{H})$  is said to be admissible if it is  $K$ -finite and if  $T|_K$  acts on  $\mathcal{H}$  by unitary operators.

A representation  $U$  of an (associative or Lie) algebra  $\mathcal{A}$  on a  $\mathbf{C}$ -vector space  $E$  is denoted  $(U, E)$ . The term  $\mathcal{A}$ -module is also used. Irreducibility for  $U$  always means algebraic irreducibility (= no proper invariant subspaces). Let  $\widehat{\mathfrak{k}}_{\mathbf{C}}$  denote the set of equivalence classes of finite dimensional simple  $\mathfrak{k}_{\mathbf{C}}$ -modules. The sum of all simple  $\mathfrak{k}_{\mathbf{C}}$ -submodules of  $E$  which are in the class  $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$  is denoted by  $E(\delta)$ .  $(U, E)$  is said  $\mathfrak{k}$ -finite if  $\dim E(\delta) < \infty$  for all  $\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}$  and if  $E = \sum_{\delta \in \widehat{\mathfrak{k}}_{\mathbf{C}}} E(\delta)$ .

Every  $K$ -finite irreducible representation  $(T, \mathcal{H})$  of  $G$  induces a  $\mathfrak{k}$ -finite irreducible representation  $(T_K, \mathcal{H}_K)$  of  $\mathfrak{U}(\mathfrak{g})$  by differentiation. If, moreover,  $\mathcal{H}$  is Hilbert and  $T$  is unitary, then  $\mathfrak{g}$  acts on  $\mathcal{H}_K$  by skew-adjoint operators:  $\langle T_K(X)\varphi, \psi \rangle = -\langle \varphi, T_K(X)\psi \rangle$  for all  $X \in \mathfrak{g}$  and all  $\varphi, \psi \in \mathcal{H}_K$ . Two  $K$ -finite representations  $(T, \mathcal{H}), (T', \mathcal{H}')$  of  $G$  are said to be *infinitesimally equivalent* if the representations  $(T_K, \mathcal{H}_K), (T'_K, \mathcal{H}'_K)$  of  $\mathfrak{U}(\mathfrak{g})$  are equivalent.

Assume  $G$  is simply connected (which is the case for  $G = \mathrm{Sp}(1, n)$ ). It is a result of Harish-Chandra ([HC1], Theorem 9; see also [W1], pp. 330–331) that if  $(U, S)$  is an algebraically irreducible  $\mathfrak{k}$ -finite representation of  $\mathfrak{U}(\mathfrak{g})$  and if  $S$  can be endowed with a positive definite Hermitian form  $\langle \cdot, \cdot \rangle$  for which  $\mathfrak{g}$  acts on  $(S, \langle \cdot, \cdot \rangle)$  via skew-adjoint operators, then there is a unique unitary irreducible representation  $\widetilde{T}$  of  $G$  on the Hilbert completion  $\widetilde{\mathcal{H}}$  of  $S$  with respect to  $\langle \cdot, \cdot \rangle$  so that  $\widetilde{\mathcal{H}}_K = S$  and  $\widetilde{T}_K = U$ . We say in this case that  $(U, S)$  – or simply  $S$  if  $U$  is understood – is *unitarizable*. If, in particular,  $(U, S) = (T_K, \mathcal{H}_K)$  for a  $K$ -finite irreducible representation  $(T, \mathcal{H})$  of  $G$ , then  $(T, \mathcal{H})$  and  $(\widetilde{T}, \widetilde{\mathcal{H}})$  are infinitesimally equivalent. The converse is also obvious: if  $(T, \mathcal{H})$  is an irreducible  $K$ -finite representation of  $G$  which is infinitesimally equivalent to a unitary Hilbert representation  $(\widetilde{T}, \widetilde{\mathcal{H}})$  of  $G$ , then  $(T_K, \mathcal{H}_K)$  is unitarizable.

As we are going to show, the  $\tau_l$ -spherical functions can be written as

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)E(\tau_l)] = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)]$$

for certain admissible irreducible Hilbert representations  $(T_{l,s}, \mathcal{H}_{l,s})$  of  $G = \operatorname{Sp}(1, n)$  satisfying  $\dim \mathcal{H}_{l,s}(\tau_l) = d_l$  (for the second equality see e.g. [HC2], Lemma 1). The positive definite  $\zeta_{l,s}$  can then be selected by applying the following theorem.

5.1. THEOREM ([Sak], Theorem 3; [B], I.4.8, p.44).  *$\zeta_{l,s}$  is positive definite if and only if  $(T_{l,s}, \mathcal{H}_{l,s})$  is infinitesimally equivalent to a unitary representation.*

Realize  $\tau_l$  as a unitary representation on a  $(2l + 1)$ -dimensional Hilbert space  $V_l$  with inner product  $\langle \cdot, \cdot \rangle_l$ . For all  $s \in \mathbf{C}$ , define a representation  $\theta_{l,s}$  of  $P = MAN$  on  $V_l$  by

$$\theta_{l,s}(ma_t n) = e^{-(s-\rho)t} \tau_l(m).$$

Consider the representation  $T'_{l,s} = \operatorname{Ind}_P^G(\theta_{l,s})$  of  $G = \operatorname{Sp}(1, n)$ : the representation space is the Hilbert completion  $\mathcal{H}'_{l,s}$  of the set of the  $C^\infty$  functions  $F: G \rightarrow V_l$  satisfying

$$F(gp) = \theta_{l,s}(p^{-1})F(g) = e^{(s-\rho)t} \tau_l(m^{-1})F(g), \quad g \in G, p = ma_t n \in P,$$

with respect to the inner product

$$(F_1, F_2)_l = \int_K \langle F_1(k), F_2(k) \rangle_l dk.$$

$G$  acts according to

$$(T'_{l,s}(g)F)(g') = F(g^{-1}g'), \quad g, g' \in G.$$

$T'_{l,s}$  is admissible, but need not be irreducible.

The following lemma is a straightforward generalization of the result in Section 16, pp.526–528, of [Go]. We therefore omit its proof.

5.2. LEMMA. *For all  $l \in \mathbf{N}/2$  and  $s \in \mathbf{C}$ , let  $E'(\tau_l)$  denote the projection of  $\mathcal{H}_{l,s}$  onto its  $K$ -isotypic subspace of type  $\tau_l$ . Then*

$$\zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E'(\tau_l)T'_{l,s}(g)].$$

The composition series structure and unitarity for the  $T'_{l,s}$  have been determined by Howe and Tan with infinitesimal methods. In [HT], the results about the  $T'_{l,s}$  are deduced from those obtained for a certain family of representations of  $\mathrm{Sp}(1, n) \times \mathbf{H}^\times$  which are equivalent to  $T'_{l,s} \otimes \tau_{l,s}$ . Here  $\mathbf{H}^\times = \mathbf{R}_+^\times \cdot \mathrm{Sp}(1)$  denotes the group of quaternionic dilations, acting on the space  $V_l$  of  $\tau_l$  according to

$$\tau_{l,s}(h) = |h|^{s-\rho} \tau_l(h/|h|), \quad h \in \mathbf{H}^\times.$$

5.3. THEOREM ([HT], Theorem 5.6 and p.58).

1.  $(\mathcal{H}'_{l,s})_K$  is equivalent as a  $\mathfrak{U}(\mathfrak{g})$ -module to  $(\mathcal{H}'_{l,-s})_K$ .
2.  $(\mathcal{H}'_{l,s})_K$  is a reducible  $\mathfrak{U}(\mathfrak{g})$ -module if and only if  $s \in \mathbf{Z}$ ,  $s \equiv 2(l-n)+1 \pmod{2}$  and  $s \notin (2l - \rho + 2, -2l + \rho - 2)$ .
3. Suppose  $(\mathcal{H}'_{l,s})_K$  irreducible. Then  $(\mathcal{H}'_{l,s})_K$  is unitarizable if and only if one of the following two cases occurs:

(a)  $s = i\nu$ ,  $\nu \in \mathbf{R}$ .

(b)  $s \in (2l - \rho + 2, -2l + \rho - 2)$ .

Case (b) corresponds to the complementary series for  $\mathrm{Sp}(1, n)$ . They exist if and only if  $2l < 2n - 1$ .

The fact that  $\tau_l$  occurs exactly once in  $T'_{l,s}|_K$  for the irreducible  $T'_{l,s}$  is known a priori ([Go], Corollary to Theorem 8, p.522; [Dei], Theorem 3). The explicit  $K$ -module decomposition of  $(\mathcal{H}'_{l,s})_K$  in [HT], pp.53–54, shows that this is actually true for all the  $T'_{l,s}$ . The  $K$ -submodule of  $(\mathcal{H}'_{l,s})_K$  equivalent to  $\tau_l$  is the only element in the “fiber of  $K$ -types” over the point  $(0, 2l)$  in Diagrams 5.10 and 5.14 of [HT]. It is contained in a unique subquotient of  $T'_{l,s}$ , which can then be located in the diagrams used to determine the unitarizability of the various subquotients ([HT], pp.25 and 30). We therefore obtain the following proposition.

5.4. PROPOSITION. Suppose  $(\mathcal{H}'_{l,s})_K$  is a reducible  $\mathfrak{U}(\mathfrak{g})$ -module and assume  $s \geq 0$ . The irreducible subquotient of  $(\mathcal{H}'_{l,s})_K$  in which  $\tau_l$  occurs is unitarizable if and only if  $s \equiv 2(l-n)+1 \pmod{2}$  and  $2l > s - \rho + 4n - 2$ . That is, if and only if  $2l \geq 2n - 1$  and  $s \in \{s_j = 2(l-n-j)+1 : j = 0, 1, \dots; s_j \geq 0\}$ .

Let  $(T_{l,s}, \mathcal{H}_{l,s})$  denote the subquotient representation of  $T'_{l,s}$  corresponding to the irreducible subquotient of  $(\mathcal{H}'_{l,s})_K$  in which  $\tau_l$  occurs. Then  $T_{l,s}$  is an admissible Hilbert representation of  $\mathrm{Sp}(1, n)$ , and  $T_{l,s}(g)v = T'_{l,s}(g)v$  for all  $v \in \mathcal{H}'_{l,s}(\tau_l)$ . Lemma 5.2 yields

5.5. COROLLARY. Let  $E(\tau_l)$  denote the projection of  $\mathcal{H}_{l,s}$  onto the  $K$ -isotypic subspace of type  $\tau_l$ . Then

$$(5.43) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)].$$

$(T_{l,s}, \mathcal{H}_{l,s})$  is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of  $(\mathcal{H}'_{l,s})_K$  is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM.  $\zeta_{l,s} = \zeta_{l,-s}$  is positive definite if and only if one of the following cases occurs:

1.  $s = i\nu$ ,  $\nu \in \mathbf{R}$ .
2. If  $2l \geq 2n - 1$ :  $\pm s = s_j := 2(l - n - j) + 1$  for integers  $j \geq 0$  so that  $s_j > 0$ . (discrete series)
3. If  $2l < 2n - 1$ :  $s \in (2l - \rho + 2, -2l + \rho - 2)$ . (complementary series)

The situation for  $s$  real and nonnegative is represented in Figure 6.1.

## 6. THE $\tau_l$ -ABEL TRANSFORM

Proposition 3.2 proves that the  $\tau_l$ -Abel transform is a  $*$ -homomorphism of  $\mathcal{D}(G; \chi_l)$  into the convolution algebra  $\mathcal{D}_+(\mathbf{R})$  consisting of the even  $C^\infty$  functions on  $\mathbf{R}$  with compact support. The main theorem of this section states that the  $\tau_l$ -Abel transform is also a bijection of  $\mathcal{D}(G; \chi_l)$  onto  $\mathcal{D}_+(\mathbf{R})$ , and gives a formula for its inverse.

Identify  $A$  with  $\mathbf{R}$  under the map  $t \mapsto a_t$ . Restriction to  $A$  then identifies  $\mathcal{D}(G; \chi_l)$  with  $\mathcal{D}_+(\mathbf{R})$ . Let  $\mathcal{D}([1, \infty))$  denote the set of the compactly supported  $C^\infty$  functions on  $[1, \infty)$  (right differentiability at 1 is considered). Define a map  $H$  by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$