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Autor: Mattner, Lutz

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PRODUCT MEASURABILITY, PARAMETER INTEGRALS, AND A FUBINI COUNTEREXAMPLE

by Lutz MATTNER

Negative results:

1) A convolution $\int g(\cdot - y)h(y)\, dy$ need not be measurable with respect to the σ-algebra generated by the translates of g.
2) There exist a Borel set $A \subset \mathbf{R}$ and two σ-finite measures μ, ν such that

$$\int\int 1_A(x+y)\, d\mu(x) d\nu(y) \neq \int\int 1_A(x+y)\, d\nu(y) d\mu(x)\,.$$

Positive result:

A function of two variables, measurable with respect to a product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ and partially measurable with respect to $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, is $\mu \otimes \nu$-almost measurable with respect to $\mathcal{A}_0 \otimes \mathcal{B}_0$, for μ, ν σ-finite measures on $\mathcal{A}$, $\mathcal{B}$.

1. INTRODUCTION

Let $(\mathcal{Y}, \mathcal{B}, \nu)$ be a σ-finite measure space, let $\mathcal{X}$ be a set, and let $f \colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ be a function with $f(x, \cdot)$ $\mathcal{B}$-measurable for every $x \in \mathcal{X}$. Then

$$F(x) := \int_{\mathcal{Y}} f(x,y)\, d\nu(y) \qquad (x \in \mathcal{X}) \tag{1}$$

defines a function $F \colon \mathcal{X} \to [0, \infty]$. Now let $\mathcal{A}_0$ be a σ-algebra on $\mathcal{X}$, and assume that $f(\cdot, y)$ is $\mathcal{A}_0$-measurable for every $y \in \mathcal{Y}$. Does it follow that F is $\mathcal{A}_0$-measurable? Surprisingly, most books on measure and integration ignore this question. Regrettably, the answer is no. Already Sierpiński (1920) provided a counterexample. His construction uses the axiom of choice and the continuum hypothesis. Without using these or similar axioms, we present below in paragraph 2.1 a simple example of a nonmeasurable F, with the right hand side of (1) being of convolution type. On the other hand, Theorem 3.1 contains a positive result, almost yielding $\mathcal{A}_0$-measurability of F under additional assumptions.

Now assume that we are also given a σ-finite measure μ on $(\mathcal{X}, \mathcal{A}_0)$. Let us further assume that the function F from (1) is $\mathcal{A}_0$-measurable and, similarly, that $G := \int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\mathcal{B}$-measurable. Does it then follow that the "Fubini identity" $\int_{\mathcal{Y}} G(y)\, d\nu(y) = \int_{\mathcal{X}} F(x)\, d\mu(x)$ holds ? Again, the answer is no, as Sierpiński (1920) remarked, essentially by specializing his construction mentioned above. This counterexample has found its way into a number of books, for example Rudin (1987) and Royden (1988), as showing that the assumption of measurability of f with respect to the product σ-algebra $\mathcal{A}_0 \otimes \mathcal{B}$ in the Fubini theorem is not superfluous. In its construction the axiom of choice is still used. The continuum hypothesis is needed only if one insists on specifying the measure spaces, for example as Lebesgue measure. That something beyond the axiom of choice is really needed in the latter case has been proved by Friedman (1980). Below we give, without using the axiom of choice or the continuum hypothesis, a simple construction of a Borel set $A \subset \mathbf{R}$ and of two σ-finite measures μ and ν, defined on suitable σ-algebras on $\mathbf{R}$, such that

$$(2) \qquad \int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y)\, d\mu(x) \right] d\nu(y) \neq \int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y)\, d\nu(y) \right] d\mu(x),$$

with both iterated integrals existing.

2. NONMEASURABILITY AND A FUBINI COUNTEREXAMPLE

2.1 A NONMEASURABLE CONVOLUTION

In this section, we show that a convolution

$$(3) \qquad F := \int_{\mathbf{R}} g(\cdot - y) h(y)\, dy,$$

with g being a nonnegative bounded Borel function and h nonnegative continuous with compact support, need not be measurable with respect to

$$(4) \qquad \mathcal{A}_0 := \sigma(\{g(\cdot - y) \,:\, y \in \mathbf{R}\}),$$

the σ-algebra generated by the translates of g. This yields in particular a counterexample to the measurability of F from (1), with $f(x,y) = g(x-y)h(y)$,