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Kapitel: 2. NONMEASURABILITY AND A FUBINI COUNTEREXAMPLE

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in paragraph 2.1 a simple example of a nonmeasurable F, with the right hand side of (1) being of convolution type. On the other hand, Theorem 3.1 contains a positive result, almost yielding $\mathcal{A}_0$-measurability of F under additional assumptions.

Now assume that we are also given a σ-finite measure μ on $(\mathcal{X}, \mathcal{A}_0)$. Let us further assume that the function F from (1) is $\mathcal{A}_0$-measurable and, similarly, that $G := \int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\mathcal{B}$-measurable. Does it then follow that the "Fubini identity" $\int_{\mathcal{Y}} G(y)\, d\nu(y) = \int_{\mathcal{X}} F(x)\, d\mu(x)$ holds? Again, the answer is no, as Sierpiński (1920) remarked, essentially by specializing his construction mentioned above. This counterexample has found its way into a number of books, for example Rudin (1987) and Royden (1988), as showing that the assumption of measurability of f with respect to the product σ-algebra $\mathcal{A}_0 \otimes \mathcal{B}$ in the Fubini theorem is not superfluous. In its construction the axiom of choice is still used. The continuum hypothesis is needed only if one insists on specifying the measure spaces, for example as Lebesgue measure. That something beyond the axiom of choice is really needed in the latter case has been proved by Friedman (1980). Below we give, without using the axiom of choice or the continuum hypothesis, a simple construction of a Borel set $A \subset \mathbf{R}$ and of two σ-finite measures μ and ν, defined on suitable σ-algebras on $\mathbf{R}$, such that

$$(2) \qquad \int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y)\, d\mu(x) \right] d\nu(y) \neq \int_{\mathbf{R}} \left[\int_{\mathbf{R}} 1_A(x+y)\, d\nu(y) \right] d\mu(x),$$

with both iterated integrals existing.

2. Nonmeasurability and a Fubini counterexample

2.1 A nonmeasurable convolution

In this section, we show that a convolution

$$(3) \qquad F := \int_{\mathbf{R}} g(\cdot - y) h(y)\, dy,$$

with g being a nonnegative bounded Borel function and h nonnegative continuous with compact support, need not be measurable with respect to

$$(4) \qquad \mathcal{A}_0 := \sigma(\{g(\cdot - y) \,:\, y \in \mathbf{R}\}),$$

the σ-algebra generated by the translates of g. This yields in particular a counterexample to the measurability of F from (1), with $f(x,y) = g(x-y)h(y)$,

$\mathcal{X} = \mathcal{Y} = \mathbf{R}$, $\mathcal{A} = \mathcal{A}_0$ from (4), $\mathcal{B} = \mathcal{B}(\mathbf{R}) :=$ Borel σ-algebra on $\mathbf{R}$, and $\nu = \lambda :=$ Lebesgue measure on $\mathcal{B}(\mathbf{R})$.

The construction becomes clearer if we first drop the nonnegativity and boundedness conditions imposed on g, for which the necessary modifications are indicated afterwards.

Remember that a set $A \subset \mathbf{R}$ is called *meager* [or of the first category] if there is a sequence of closed and nowhere dense sets $F_n \subset \mathbf{R}$ with $A \subset \bigcup_{n\in\mathbf{N}} F_n$. Correspondingly, a set $B \subset \mathbf{R}$ is called *comeager* if its complement B^c is meager, which is equivalent to the existence of a sequence of dense open sets $G_n \subset \mathbf{R}$ with $B \supset \bigcap_{n\in\mathbf{N}} G_n$. It is easily checked that

(5) $$\mathcal{A} := \{A \in \mathcal{B}(\mathbf{R}) : A \text{ meager or comeager}\}$$

is a σ-algebra on $\mathbf{R}$. By Baire's theorem, every comeager set is dense in $\mathbf{R}$. [We have claimed in the introduction not to use the axiom of choice in constructing this example and the one in 2.2. So we have to note here that we are applying Baire's theorem only in $\mathbf{R}$, a *separable* complete metric space, where no form of the axiom of choice is needed in its proof. Compare Oxtoby (1980), page 95.] It follows that, for example, the set $[0, \infty[$ is neither meager nor comeager. Hence we surely have the strict inclusion

(6) $$\mathcal{A} \subsetneq \mathcal{B}(\mathbf{R}).$$

Now choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$, for example as in Oxtoby (1980), pages 4–5. Put

(7) $$g(x) := x\,1_A(x) \qquad (x \in \mathbf{R}),$$

(8) $$h(x) := (1 - |x|)_+ \qquad (x \in \mathbf{R}),$$

and define F as in (3) and $\mathcal{A}_0$ as in (4). Then

(9) $$\mathcal{A}_0 \subset \mathcal{A},$$

because every $g(\cdot - y)$ is Borel and vanishes on the comeager set $(y+A)^c$, and is hence $\mathcal{A}$-measurable. On the other hand, since $\lambda(A^c) = 0$ and $\int h\,dy = 1$, $\int yh(y)\,dy = 0$,

(10) $$F(x) = \int_{\mathbf{R}} (x - y)h(y)\,dy = x \qquad (x \in \mathbf{R}).$$

Hence $\sigma(F)$, the σ-algebra generated by F, is just $\mathcal{B}(\mathbf{R})$, and by (6), (9) it follows that F is not $\mathcal{A}_0$-measurable.

To obtain that same conclusion for a nonnegative and bounded g, we may replace g from (7) by $g(x) := (\pi/2 + \arctan x)1_A(x)$. Instead of calculating F explicitly, we then argue that F is still strictly increasing, and this suffices to deduce that $\sigma(F) = \mathcal{B}(\mathbf{R})$.

2.2 A Fubini counterexample

In this section, we give an example of (2). Let $\mathcal{A}$ be as in (5) and define $\mu|_{\mathcal{A}}$ by

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is meager}, \\ 1 & \text{if } A \text{ is comeager}. \end{cases} \tag{11}$$

This is possible, since no set $A \subset \mathbf{R}$ is simultaneously meager and comeager, for otherwise $\varnothing = A \cap A^c$ would be comeager, in contradiction to Baire's theorem. It is easy to check that μ is a probability measure on $(\mathbf{R}, \mathcal{A})$. Let again $\nu := \lambda :=$ Lebesgue measure on $\mathcal{B} := \mathcal{B}(\mathbf{R})$, and choose $A \in \mathcal{A}$ meager with $\lambda(A^c) = 0$. Then $1_A(\cdot + y)$ is $\mathcal{A}$-measurable with

$$\int_{\mathbf{R}} 1_A(x+y)\, d\mu(x) = \mu(A - y) = 0 \qquad (y \in \mathbf{R}).$$

On the other hand, we have

$$\int_{\mathbf{R}} 1_A(x+y)\, d\nu(y) = \lambda(A - x) = \infty \qquad (x \in \mathbf{R}).$$

Hence (2) is obviously true in this case.

3. Measurability

Here is a positive result, having a certain measurability property of F from (1) among its conclusions. An application of this occurs in Mattner (1999).

3.1. Theorem. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces, let $f\colon \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ be a function measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$, and put*

$$\begin{aligned} \mathcal{A}_0 &:= \sigma(\{f(\cdot, y) : y \in \mathcal{Y}\}), \\ \mathcal{B}_0 &:= \sigma(\{f(x, \cdot) : x \in \mathcal{X}\}), \\ \overline{\mathcal{A}}_0 &:= \{A \in \mathcal{A} : \exists A_0 \in \mathcal{A}_0 \text{ with } A = A_0 \quad [\mu]\}, \\ \overline{\mathcal{B}}_0 &:= \{B \in \mathcal{B} : \exists B_0 \in \mathcal{A}_0 \text{ with } B = B_0 \quad [\nu]\}, \\ \overline{\mathcal{A}_0 \otimes \mathcal{B}_0} &:= \{C \in \mathcal{A} \otimes \mathcal{B} : \exists C_0 \in \mathcal{A}_0 \otimes \mathcal{B}_0 \text{ with } C = C_0 \quad [\mu \otimes \nu]\}. \end{aligned}$$

Then f is $\overline{\mathcal{A}_0 \otimes \mathcal{B}_0}$-measurable, $\int_{\mathcal{Y}} f(\cdot, y)\, d\nu(y)$ is $\overline{\mathcal{A}}_0$-measurable, and $\int_{\mathcal{X}} f(x, \cdot)\, d\mu(x)$ is $\overline{\mathcal{B}}_0$-measurable.