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A GENERALIZED FØLNER CONDITION AND THE NORMS OF RANDOM WALK OPERATORS ON GROUPS

by Andrzej ŻUK

ABSTRACT. We prove a generalized Følner condition. We present a method of computing and estimating the norms of random walk operators on groups and graphs. We give explicit computations in several cases.

1. INTRODUCTION

Let us consider a pair (Γ, S), where Γ is a finitely generated group and S is a finite, symmetric set of generators (symmetric means $S = S^{-1}$).

For a finite subset $A \subset \Gamma$ we define its *boundary*

$$\partial A = \{\gamma \in A; \text{ there exists } s \in S \text{ such that } \gamma s \notin A\}.$$

A *Følner sequence* is a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of Γ such that the cardinality of the boundary ∂A_n of the set A_n divided by the cardinality of A_n tends to zero, i.e.

$$\frac{\#\partial A_n}{\#A_n} \rightarrow_{n\rightarrow\infty} 0.$$

Følner proved in [4] that the existence of such a sequence is equivalent to amenability of the group Γ.

One can associate with the pair (Γ, S) the *simple random walk operator* $P: l^2(\Gamma) \rightarrow l^2(\Gamma)$:

$$Pf(\gamma) = \frac{1}{\#S} \sum_{s\in S} f(\gamma s) \quad \text{for } f \in l^2(\Gamma).$$

Let $||P||$ be the operator norm of P acting on $l^2(\Gamma)$. In [8] Kesten proved:

THEOREM 1 (Kesten). *The following conditions are equivalent:*

(1) $||P|| = 1$.

(2) *The group* Γ *is amenable, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* Γ *satisfying the Følner condition.*

In the next section we will prove a generalization of this result (Theorems 2 and 3), showing that equalities of the form $||P|| = \lambda$, with $0 < \lambda \leq 1$, are equivalent to appropriate Følner-like conditions. Section 3 is devoted to some remarks concerning this generalization. In Section 4 we use the generalized Følner condition to compute the norms of some random walk operators and in Section 5, using the same ideas, we obtain some lower bounds for the random walk operators on graphs.

After completion of this work, we learned that some versions of a generalized Følner condition were obtained recently by S. Popa [12].

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2. THE GENERALIZED FØLNER CONDITION

Let us consider a measurable space $(X, \mathcal{F})$. On this space we consider a *Markov transition kernel* $P(\cdot, \cdot)$, i.e. for any $x \in X$, $P(x, \cdot)$ is a probability measure on $(X, \mathcal{F})$ and $P(\cdot, A)$ is a measurable function on $(X, \mathcal{F})$ for every $A \in \mathcal{F}$.

Let μ be a σ-finite measure on the space $(X, \mathcal{F})$. For any measurable subset $A \subset X$ we define its measure $|A|$ and the measure $|\partial A|$ of its boundary ∂A as follows:

$$|A| = \mu(A),$$

$$|\partial A| = \int_{\{x \in A\}} \int_{\{y \in A^c\}} P(x, dy) d\mu(x).$$