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For the above values, f is an eigenfunction of the operator P and satisfies the generalized Følner condition. By Theorem 3 the norm of the random walk operator on $\mathbf{Z}_2 \star \mathbf{Z}_4$ with the generating subset as defined before is then equal to

$$\|P\| = \frac{\sqrt{33} + 7}{\sqrt{\sqrt{33} - 1}} \approx 0.98.$$

4.2.2 GENERAL CASE

The idea presented for $\mathbf{Z}_2 \star \mathbf{Z}_4$ can be used in the general case for $\mathbf{Z}_n \star \mathbf{Z}_m$. As the solution involves roots of some polynomial of degree nm , we will not give details.

4.3 MEAN OPERATOR ON THE HYPERBOLIC PLANE

Let us consider the hyperbolic upper half-plane $H = \{z = x + iy \in \mathbf{C}; x \in \mathbf{R}, y > 0\}$ with a Riemannian metric $d_{Hz} = \frac{\sqrt{dx^2 + dy^2}}{y}$ which gives rise to the measure $\mu_H = \frac{dx dy}{y^2}$. We consider the operator P ,

$$Pf(z_0) = \int_{|z-z_0|=R} f(z) dm_R(z),$$

where dm_R is a uniform probability measure on a hyperbolic circle of radius R . We want to compute the norm of the operator P acting on $L^2(H, d_{Hz})$.

First of all let us remark that the function:

$$(11) \quad f(z) = \sqrt{\operatorname{Im}(z)},$$

is an eigenfunction of P . An easy way to see this is to note that P commutes with isometries of H and that the isometries consisting of horizontal translations and homotheties act transitively on H . The effect of these on the function f is that they just multiply it by a constant.

Now we would like to show that one can find a Følner sequence with respect to the function f . Let us consider a sequence $\{A_n\}_{n=1}^{\infty}$ of rectangles (in the Euclidean sense) in H :

$$A_n = \{z \in H; e^{-n} \leq \operatorname{Im}(z) \leq 1, 0 \leq \operatorname{Re}(z) \leq n\}.$$

It is easy to see that the measure $|\partial A_n|$ of the boundary of A_n is bounded by the measure of the following set B_n (see Figure 5):

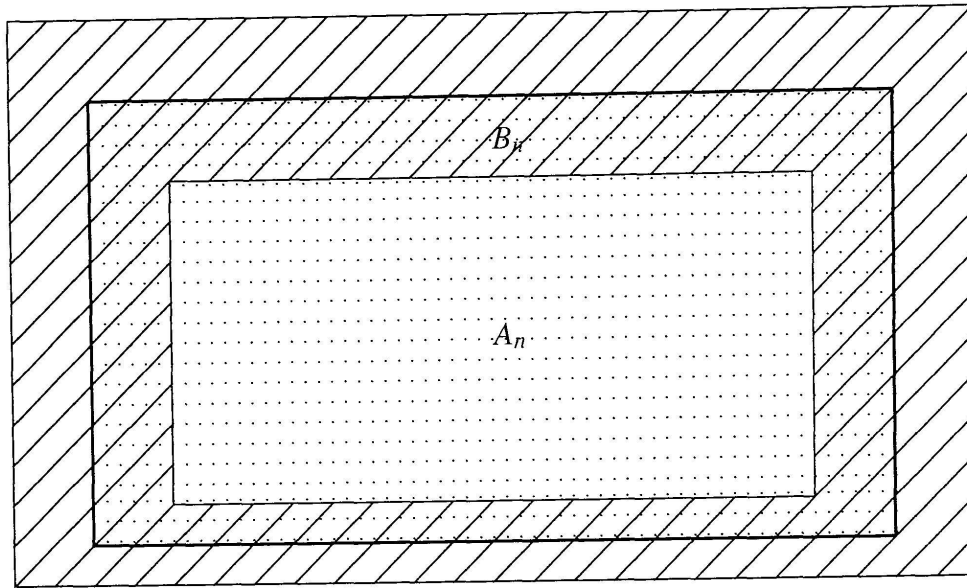


FIGURE 5
Sets A_n and B_n

$$\begin{aligned}
 B_n = & \{z \in H; -R \leq \operatorname{Re}(z) \leq R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R+n \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^{-n+R} \geq \operatorname{Im}(z) \geq e^{-n-R}\}.
 \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that $\{A_n\}_{n=1}^\infty$ is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H, d_{Hz}) \rightarrow L^2(H, d_{Hz})} = \int_{|z-i|=R} \sqrt{\operatorname{Im}(z)} \, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g: F \rightarrow G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F , and where γ_1 is an element of F . The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where