1. Cubical complexes

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The paper is organized as follows. In Section 1 we recall basic definitions and facts related to cubical complexes, prove some criteria for foldability and discuss nonpositive curvature. In Section 2 we recall some constructions and examples of foldable cubical complexes. In Section 3, we introduce hyperplanes in cubical complexes as in [NR]. Foldability then leads to systems of disjoint hyperplanes and their "dual trees" which will accomplish the proof of Theorem 1. In Section 4 we investigate the induced actions on the dual trees and obtain the proof of Theorem 2. In Section 5 we develop the idea of parallel transport in cubical manifolds and use it to prove Theorem 3.

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1. Cubical complexes

In this section we briefly recall basic notions and facts related to cubical complexes.

Cubical complexes and cubical metric

A *cell* P is the convex hull of a finite set of points in a real vector space. Faces of P are then well defined, and they are also cells (see e.g. [Br]). The set $\mathcal{P}$ of faces of P is partially ordered by inclusion and called the *poset* of P. Two cells are *combinatorially equivalent* if their posets are isomorphic. For example, every convex quadrilateral polygon is combinatorially equivalent to the unit square. An isomorphism of posets induces a bijection between sets of barycenters of faces and thus determines a piecewise linear homeomorphism between two cells. We call such a homeomorphism a *realization* of a combinatorial equivalence.

A *cell complex* is a collection X of cells which are glued by realizations of combinatorial equivalences along faces. We also assume that different faces of the same cell are not identified and that the intersection of different cells is either empty or consists of one cell. These latter assumptions are not essential, but they simplify the exposition considerably. However, we do not require that X is locally finite, so that, if not explicitely stated otherwise, a vertex in X may belong to infinitely many distinct cells.

We say that a cell complex X is *simplicial* if the cells of X are simplices. Because of our assumptions on the glueing of faces, this coincides with the standard terminology. We say that X is *cubical* if the cells of X are combinatorially equivalent to cubes.

Let X be a cubical complex. Any combinatorial equivalence of a Euclidean unit cube is an isometry, hence any cell P in X is endowed with a canonical metric d_P which makes it isometric to the Euclidean unit cube. This allows to measure the lengths of finite polygonal paths in X. Let d be the associated length pseudometric on X. Then d is actually a metric and turns X into a complete geodesic space, see [B1]. We call d the *cubical metric.*

RESIDUES AND LINKS

For a cell complex X and a cell P in X, the *residue* of P, denoted $\operatorname{res} P$, consists of all cells of X containing P. The residue of a cell is a closed subcomplex of X.

Let X be a cell complex. If P and Q are cells in X with $Q \in \operatorname{res} P$, then the poset consisting of all faces R of Q with $P \neq R \supset P$ is a poset of a cell Q_P, well defined up to combinatorial equivalence and of dimension $\dim Q_P = \dim Q - \dim P$. We define the *link* X_P of a cell complex X at a cell P as the collection of the cells Q_P, one for each cell Q in $\operatorname{res} P$, with the natural identifications of faces induced from X.

We will need residues and links only in the case when X is simplicial or cubical. In both cases, the links are simplicial. In the simplicial case, the residue of a simplex P of X is naturally homeomorphic to the simplicial join of P and X_P, in the cubical case to the cubical cone over X_P times P. (See the subsection on right angled Coxeter complexes in Section 2 for the definition of the cubical cone.)

GALLERIES AND CHAMBER COMPLEXES

An n-dimensional cell complex X is called *dimensionally homogeneous* if each cell of X is contained in an n-dimensional cell. If X is dimensionally homogeneous, then the top-dimensional cells of X will be called *chambers*, the cells of codimension 1 *panels.*

A special case occurs when X is homeomorphic to a manifold. In this case we say that X is a *cellular manifold*, speaking also about simplicial or cubical manifolds if all the chambers are simplices or cubes respectively.

Let X be a dimensionally homogeneous cell complex. A *gallery* in X is a sequence of chambers where any two consecutive chambers have a panel in common. We say that X is *gallery connected* if any two chambers of X can be connected by a gallery. If X is gallery connected, then we say that X is a *chamber complex.* Tits buildings and connected cellular manifolds are chamber complexes.

We say that X is *locally gallery connected* if the link of each cell of X of codimension greater than 1 is gallery connected. If X is connected and locally gallery connected, then X is gallery connected and hence a chamber complex.

FOLDINGS

A *folding* of an n-dimensional simplicial (respectively cubical) complex X is a combinatorial map of X onto an n-simplex (respectively n-cube) which is injective on each cell of X. A *folded simplicial* (respectively *folded cubical*) *complex* is a simplicial (respectively cubical) complex together with a folding.

A simplicial (respectively cubical) complex X is *foldable* if it admits a folding, *locally foldable* if the link of each cell of X is foldable. The following lemma gives a criterion for foldability of a cubical complex in terms of local properties.

LEMMA 1.1. *Let X be a simply connected cubical chamber complex of dimension n. If X is locally gallery connected and locally foldable, then X is foldable and a folding of X is unique up to an automorphism of the n-cube.*

Proof. We observe that foldability (respectively gallery connectedness) holds for the residue of a cell P of X if and only if it holds for the link X_P. Therefore the assumptions of the lemma imply that all residues in X are foldable and gallery connected.

A curve $c\colon [0,1] \to X$ is called *generic* if it crosses the codimension one skeleton of X at finitely many points. We will call such points *singular*. Since X is dimensionally homogeneous, generic curves are dense in the space of all curves in X.

Let c be a generic curve connecting interior points p and q of chambers P and Q of X. Define an isomorphism $f_c\colon Q \to P$ as follows. If c has no singular point we set $f_c = \mathrm{id}_P$. If c has one singular point, let R be a cell of X containing this singular point in its interior. Then the whole curve c is contained in the residue of R. Since $\operatorname{res} R$ is foldable, there exists a folding map $f\colon \operatorname{res} R \to P$ which extends id_P; since $\operatorname{res} R$ is gallery connected, f is unique. We set $f_c := f|_Q$. Finally, if c has more than one singular point, we cut c into a sequence c_i of curves, each of which has exactly one singular point in its interior, and define f_c to be the composition of the isomorphisms f_{c_i}.

We show now that $f_c = \mathrm{id}$ for each closed generic curve at p. Let c be such a curve. Since X is simply connected, c can be contracted to p. Such a contraction can be chosen to be generic, that is, it consists of generic curves only and singular points appear or disappear only at a finite number of times during the contraction. At each such time, c can be cut into finitely many pieces such that each piece is contained in the residue of a cell and such that the appearance or disappearance of singular points occurs in (some of) the pieces. Since residues are gallery connected and foldable, we conclude that f_c remains unchanged during the contraction. Now $f_p = \mathrm{id}$ for the point curve p, hence $f_c = \mathrm{id}$.

Fix a chamber P in X and an interior point p of P. For each other chamber Q of X choose an interior point $q \in Q$ and a generic curve c connecting p with q. Define a map $F: X \to P$ by $F|_Q = f_c$. The above considerations show that F is well defined, hence F is a folding of X. This proves the first assertion of the lemma.

The remaining assertion that the folding is unique up to an automorphism of the n-cube follows immediately from gallery connectedness of X. □

LEMMA 1.2. *Let X be a simply connected cubical chamber complex of dimension n. Suppose that*

(1) *the links at cells of X of codimension > 2 are simply connected;*

(2) *the links at the cells of X of codimension $= 2$ are connected bipartite graphs.*

Then X is foldable, and a folding of X is unique up to an automorphism of the n-cube.

Proof. For the purpose of this proof a curve in X is called *generic* if it misses the skeleton of codimension 2 and crosses the cells of codimension 1 transversally (note that this notion here is slightly different from the one in the proof of the previous lemma). It is clear that any two points in the interior of some chambers of X can be connected by a generic curve. If such a curve is closed, it can be contracted to a point by a contraction that misses the skeleton of codimension 3 and crosses the higher dimensional skeleta transversally.

Now we repeat the arguments of the proof of Lemma 1.1 taking only the residues of cells of codimension 2 into account. These residues consist of chambers arranged according to the corresponding links. Because the links are bipartite graphs, the residues are foldable. □

COROLLARY 1.3. *Let X be a simply connected cubical manifold of dimension n with the property that the number of chambers adjacent to each face of codimension 2 in X is even. Then X is foldable, and a folding of X is unique up to an automorphism of the n-cube.* □

NONPOSITIVE CURVATURE

We will need some elementary facts from the theory of spaces with upper curvature bounds. The main reference is [Ba].

Let (X, d) be a metric space. A curve in X is called a *geodesic* if it has constant speed and realizes the distance locally. We say that X is *geodesic* if any two points of X can be connected by a minimal geodesic. From now on we assume that X is a complete geodesic space.

Let $\kappa \in \mathbf{R}$, and let M^2_κ be the model surface of constant Gauss curvature κ. Denote by $D(\kappa)$ the diameter of M^2_κ. We say that our geodesic space X is a *CAT*(κ)-space if any geodesic triangle in X with minimal sides and of perimeter $< D(\kappa)$ is not thicker than its comparison triangle in M^2_κ. We say that X has curvature $\leq \kappa$ if any point of X has a neighborhood that is *CAT*(κ) with respect to the induced metric.

For nonpositively curved spaces, that is, spaces with upper curvature bound 0, there is the following extension of the Hadamard-Cartan Theorem.

THEOREM 1.4 (Gromov [Gr], Alexander-Bishop [AB]). *Let X be a simply connected, complete geodesic space of nonpositive curvature. Then geodesic triangles in X are not thicker than their corresponding comparison triangles in the Euclidean plane. In particular,*

(1) *for any two points $x, y \in X$, there is a unique geodesic $\sigma_{xy}\colon [0,1] \to X$ from x to y and σ_{xy} depends continuously on x and y;*
(2) *locally convex subsets of X are globally convex;*
(3) *X is contractible.*

We say that a cubical complex is *nonpositively curved* if it is nonpositively curved with respect to the cubical metric. The lemma below presents a necessary and sufficient condition for a cubical complex to be nonpositively curved in terms of its combinatorics.

A simplicial complex X is a *flag complex* if each set of vertices of X, in which any two vertices are connected by an edge, spans a simplex of X.

LEMMA 1.5 (Gromov [Gr]). *A cubical complex is nonpositively curved if and only if the link X_v at each vertex v of X is a flag complex.*

REMARK 1.6. If X is a simply connected nonpositively curved cubical complex, then the restriction of the cubical metric to any of its cells coincides with the standard Euclidean metric on the cell.