COUNTING PATHS IN GRAPHS

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COUNTING PATHS IN GRAPHS

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ABSTRACT. We give a simple combinatorial proof of a formula that extends a result by Grigorchuk [Gri78a, Gri78b] relating cogrowth and spectral radius of random walks. Our main result is an explicit equation determining the number of 'bumps' on paths in a graph: in a d-regular (not necessarily transitive) non-oriented graph let the series $G(t)$ count all paths between two fixed points weighted by their length t^{length}, and $F(u,t)$ count the same paths, weighted as $u^{\text{number of bumps}}t^{\text{length}}$. Then one has

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2} .$$

We then derive the circuit series of 'free products' and 'direct products' of graphs. We also obtain a generalized form of the Ihara-Selberg zeta function [Bas92, FZ98].

1. INTRODUCTION

Let $\Gamma = \mathbf{F}_S/N$ be a group generated by a finite set S, where $\mathbf{F}_S$ denotes the free group on S. Let f_n be the number of elements of the normal subgroup N of $\mathbf{F}_S$ whose minimal representation as words in $S \cup S^{-1}$ has length n; let g_n be the number of (not necessarily reduced) words of length n in $S \cup S^{-1}$ that evaluate to 1 in Γ; and let $d = |S \cup S^{-1}| = 2|S|$. The numbers

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{f_n} , \qquad \nu = \frac{1}{d} \limsup_{n\to\infty} \sqrt[n]{g_n}$$

are called the *cogrowth* and *spectral radius* of (Γ, S). The Grigorchuk Formula [Gri78b] states that

$$\nu = \begin{cases} \frac{\sqrt{d-1}}{d}\left(\frac{\alpha}{\sqrt{d-1}} + \frac{\sqrt{d-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{d-1} , \\ \frac{2\sqrt{d-1}}{d} & \text{else} . \end{cases} \tag{1.1}$$

We generalize this result to a somewhat more general setting: we replace the group Γ by a regular graph $\mathcal{X}$, i.e. a graph with the same number of edges at each vertex. Fix a vertex $\star$ of $\mathcal{X}$; let g_n be the number of circuits (closed sequences of edges) of length n at $\star$ and let f_n be the number of circuits of length n at $\star$ with no backtracking (no edge followed twice consecutively). Then the same equation holds between the growth rates of f_n and g_n.

To a group Γ with fixed generating set one associates its Cayley graph $\mathcal{X}$ (see Subsection 3.1). $\mathcal{X}$ is a d-regular graph with distinguished vertex $\star = 1$; paths starting at $\star$ in $\mathcal{X}$ are in one-to-one correspondence with words in $S \cup S^{-1}$, and paths starting at $\star$ with no backtracking are in one-to-one correspondence with elements of $\mathbf{F}_S$. A circuit at $\star$ in $\mathcal{X}$ is then precisely a word evaluating to 1 in Γ, and a circuit without backtracking represents precisely one element of N. In this sense results on graphs generalize results on groups. The converse would not be true: there are even graphs with a vertex-transitive automorphism group that are not the Cayley graph of a group [Pas93].

Even more generally, we will show that, rather than counting circuits and proper circuits (those without backtracking) at a fixed vertex, we can count paths and proper paths between two fixed vertices and obtain the same formula relating their growth rates.

These relations between growth rates are consequences of a stronger result, expressed in terms of generating functions. Define the formal power series

$$F(t) = \sum_{n=0}^{\infty} f_n t^n, \qquad G(t) = \sum_{n=0}^{\infty} g_n t^n .$$

Then assuming $\mathcal{X}$ is d-regular we have

$$\frac{F(t)}{1-t^2} = \frac{G\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} . \tag{1.2}$$

This equation relates F and G, and so relates *a fortiori* their radii of convergence, which are $1/\alpha$ and $1/(d\nu)$. We re-obtain thus the Grigorchuk Formula.

Finally, rather than counting paths and proper paths between two fixed vertices, we can count, for each $m \geq 0$, the number of paths with m backtrackings, i.e. with m occurrences of an edge followed twice in a row. Letting $f_{m,n}$ be the number of paths of length n with m backtrackings, consider the two-variable formal power series

$$F(u,t) = \sum_{m,n=0}^{\infty} f_{m,n}\, u^m t^n .$$

Note that $F(0,t) = F(t)$ and $F(1,t) = G(t)$. The following equation now holds:

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}.$$

Setting $u = 1$ in this equation reduces it to (1.2).

A generalization of the Grigorchuk Formula in a completely different direction can be attempted: consider again a finitely generated group Γ, and an exact sequence

$$1 \longrightarrow \Xi \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow 1,$$

where this time Π is not necessarily free. Assume Π is generated as a monoid by a finite set S. Let again g_n be the number of words of length n in Π evaluating to 1 in Γ, and let f_n be the number of elements of Ξ whose minimal-length representation as a word in S has length n. Is there again a relation between the f_n and the g_n? In Section 8 we derive such a relation when Π is the modular group $\mathbf{PSL}_2(\mathbf{Z})$.

Again there is a combinatorial counterpart; rather than considering graphs one considers a locally finite cellular complex $\mathcal{K}$ such that all vertices have isomorphic neighbourhoods. As before, g_n counts the number of paths of length n in the 1-skeleton of $\mathcal{K}$ between two fixed vertices; and f_n counts elements of the fundamental groupoid, i.e. homotopy classes of paths, between two fixed vertices whose minimal-length representation as a path in the 1-skeleton of $\mathcal{K}$ has length n. We obtain a relation between these numbers when $\mathcal{K}$ consists solely of triangles and arcs, with no two triangles nor two arcs meeting; these are precisely the complexes associated with quotients of the modular group.

The original motivation for our research was the study of cogrowth in group theory [Gri78a]; however, as it turned out, the more general problem in graph theory has applications to other domains of mathematics, like the Ihara-Selberg zeta function and its evaluation by Hyman Bass [Bas92].

2. Main result

Let $\mathcal{X}$ be a graph, that may have multiple edges and loops. We make the following typographical convention for the power series that will appear: a series in the formal variable t is written $G(t)$, or G for short, and $G(x)$ refers to the series G with x substituted for t. Functions are written on the right, with $(x)f$ or x^f denoting f evaluated at x.

We start by the precise definition of graph we will use:

DEFINITION 2.1 (Graphs). A *graph* $\mathcal{X}$ is a pair of sets $\mathcal{X} = (V, E)$ and maps

$$\alpha: E \to V, \qquad \omega: E \to V, \qquad \bar{\cdot}: E \to E$$

satisfying

$$\bar{\bar{e}} = e, \qquad \bar{e}^\alpha = e^\omega.$$

The graph $\mathcal{X}$ is said to be *finite* if both $E(\mathcal{X})$ and $V(\mathcal{X})$ are finite sets.

A *graph morphism* $\phi: \mathcal{G} \to \mathcal{H}$ is a pair of maps $(V(\phi), E(\phi))$ with $V(\phi): V(\mathcal{G}) \to V(\mathcal{H})$ and $E(\phi): E(\mathcal{G}) \to E(\mathcal{H})$ satisfying

$$\overline{eE(\phi)} = \bar{e}\, E(\phi), \qquad e^\alpha V(\phi) = (eE(\phi))^\alpha.$$

Given an edge $e \in E(\mathcal{X})$, we call e^α and e^ω e's *source* and *destination*, respectively. We say two vertices x, y are *adjacent*, and write $x \sim y$, if they are connected by an edge, i.e. if there exists an $e \in E(\mathcal{X})$ with $e^\alpha = x$ and $e^\omega = y$. We say two edges e, f are *consecutive* if $e^\omega = f^\alpha$. A *loop* is an edge e with $e^\alpha = e^\omega$.

The *degree* $\deg(x)$ of a vertex x is the number of incident edges:

$$\deg(x) = \#\{e \in E(\mathcal{X}) \mid e^\alpha = x\} = \#\{e \in E(\mathcal{X}) \mid e^\omega = x\}.$$

If $\deg(x)$ is finite for all x, we say $\mathcal{X}$ is *locally finite.* If $\deg(x) = d$ for all vertices x, we say $\mathcal{X}$ is *d-regular.*

Note that the involution $e \mapsto \bar{e}$ may have fixed points. Even though the edges of $\mathcal{X}$ are individually oriented, the graph $\mathcal{X}$ itself should be viewed as an non-oriented graph. In case $\bar{\cdot}$ has no fixed point, $\mathcal{X}$ can be viewed as a geometric graph.

DEFINITION 2.2 (Paths). A *path* in $\mathcal{X}$ is a sequence π,

$$\pi = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$$

of edges and vertices of $\mathcal{X}$, with $e_i^\alpha = v_{i-1}$ and $e_i^\omega = v_i$ for all $i \in \{1, \dots, n\}$ and $n \geq 0$. The *length* of the path π is the number n of edges in π. The *start* of the path π is $\pi^\alpha = v_0$, and its *end* is $\pi^\omega = v_n$. If $\pi^\alpha = \pi^\omega$, the path π is called a *circuit* at π^α. In most cases, we will omit the v_i from the description of paths; they are necessary only if $|\pi| = 0$, in which case a starting vertex must be specified. We extend the involution $\bar{\cdot}$ from edges to paths by setting

$$\bar{\pi} = (v_n, \overline{e_n}, \dots, v_1, \overline{e_1}, v_0)$$

(note that $\overline{\pi}$ is a path from π^ω to π^α).

We denote by $E^*(\mathcal{X})$ the set of paths, with a partially defined multiplication given by concatenation: if π and ρ be two paths with $\pi^\omega = \rho^\alpha$, their *product* is defined as $\pi\rho = (\pi_1, \dots, \pi_{|\pi|}, \rho_1, \dots, \rho_{|\rho|})$. For two vertices $x, y \in V(\mathcal{X})$ we denote by $[x, y]$ the set of paths from x to y. We turn $V(\mathcal{X})$ into a metric space by defining for vertices $x, y \in V(\mathcal{X})$ their *distance*

$$\delta(x, y) = \min_{\pi \in [x,y]} |\pi| .$$

The *ball* of radius n at a vertex $x \in V(\mathcal{X})$ is the subgraph $\mathcal{B}(x, n)$ of $\mathcal{X}$ with vertex set

$$V(\mathcal{B}(x, n)) = \{y \in V(\mathcal{X}) \, : \, \delta(x, y) \leq n\}$$

and edge set

$$E(\mathcal{B}(x, n)) = \{e \in E(\mathcal{X}) \, : \, e^\alpha \in V(\mathcal{B}(x, n))\} .$$

We define $\alpha_{\mathcal{B}}(e) = \alpha(e)$,

$$\overline{e}^{\mathcal{B}} = \begin{cases} \overline{e} & \text{if } e^\omega \in E(\mathcal{B}) \\ e & \text{else ,} \end{cases}$$

and $\omega(e) = \alpha(\overline{e})$.

This definition amounts to "wrapping around disconnected edges". It has the advantage of preserving the degrees of vertices.

DEFINITION 2.3 (Bumps, Labellings). We say a path π has a *bump at* i if $\pi_i = \overline{\pi_{i+1}}$; if the location of the bump is unimportant we will just say π has a bump. The *bump count* $\mathrm{bc}(\pi)$ of a path π is the number of bumps in π. A *proper path* in $\mathcal{X}$ is a path π with no bumps.

Let $\mathbf{k}$ be a ring. A $\mathbf{k}$*-labelling* of the graph $\mathcal{X}$ is a map

$$\ell \colon E(\mathcal{X}) \to \mathbf{k} .$$

The simplest examples of labellings are:

- the *trivial labelling*, given by $\mathbf{k} = \mathbf{Z}$ and $e^\ell = 1$ for all $e \in E(\mathcal{X})$;
- the *length labelling*, given by $\mathbf{k} = \mathbf{Z}[[t]]$ and $e^\ell = t$ for all $e \in E(\mathcal{X})$.

A $\mathbf{k}$-labelling ℓ of $\mathcal{X}$ induces a map, still written $\ell\colon E^*(\mathcal{X}) \to \mathbf{k}$, by setting

$$\pi^\ell = \pi_1^\ell \pi_2^\ell \cdot \ldots \cdot \pi_n^\ell .$$

The labelling $\ell\colon E^*(\mathcal{X}) \to \mathbf{k}$ is *complete* [Eil74, § VI.2] if for any vertex x of $\mathcal{X}$ and any set A of paths in $\mathcal{X}$ starting at x there is an element $(A)\Sigma$ of $\mathbf{k}$, and this function Σ satisfies

$$(\{\pi\})\Sigma = \pi^\ell , \qquad (A \sqcup B)\Sigma = (A)\Sigma + (B)\Sigma$$

for all paths π and disjoint sets A and B ($\sqcup$ denotes disjoint union). If A is infinite, it is customary, though abusive, to write $(A)\Sigma$ as $\sum_{\pi\in A} \pi^\ell$.

If $\mathbf{k}$ is a topological ring ($\mathbf{R}$, $\mathbf{C}$, the formal power series ring $\mathbf{Z}[[t]]$, etc.), completion of ℓ implies that $\pi^\ell \to 0$ when $|\pi| \to \infty$, but the converse does not hold. The completeness condition becomes that

$$(A)\Sigma = \lim_{B\subset A,\ |B|<\infty} \sum_{\pi\in B} \pi^\ell$$

be a well-defined element of $\mathbf{k}$ for all A; i.e., the limit exists. Generally, we define the following topology on $\mathbf{k}$: a sequence $(A_i)\Sigma \in \mathbf{k}$ converges to 0 if and only if $\min_{\pi\in A_i} |\pi|$ tends to infinity.

In the sequel of this paper all labellings will be assumed to be complete. The length labelling is complete for locally finite graphs; more generally, ℓ is complete when $\mathbf{k}$ is a discretely valued ring, e^ℓ has a positive valuation for all edges e, and $\mathcal{X}$ is locally finite. An arbitrary ring $\mathbf{k}$ may be embedded in $\mathbf{k}' = \mathbf{k}[[t]]$, where t has valuation 1 and $\mathbf{k}$ has valuation 0; if $\ell\colon E(\mathcal{X}) \to \mathbf{k}$ is a labelling, we define $\ell'\colon E(\mathcal{X}) \to \mathbf{k}'$ by $e^{\ell'} = te^\ell$; and ℓ' will be complete as soon as $\mathcal{X}$ is locally finite. In particular the length labelling is obtained from the trivial labelling through this construction. In all the examples we consider the labelling is defined in this manner.

Throughout the paper we shall assume a graph $\mathcal{X}$ and two vertices $\star, \dagger \in V(\mathcal{X})$ have been fixed. We wish to enumerate the paths, counting their number of bumps, from $\star$ to $\dagger$ in $\mathcal{X}$. For a given complete edge-labelling ℓ, consider the series

$$\mathfrak{G}(\ell) = \sum_{\pi\in[\star,\dagger]} \pi^\ell \in \mathbf{k} , \qquad \mathfrak{F}(\ell) = \sum_{\pi\in[\star,\dagger]} u^{\mathrm{bc}(\pi)}\pi^\ell \in \mathbf{k}[[u]] .$$

Note that in general $\mathfrak{G}$ and $\mathfrak{F}$ also depend on the choice of $\star$ and $\dagger$.

For all vertices $x \in V(\mathcal{X})$ let

$$K_x = \Big(1 - \sum_{e \in E(\mathcal{X}) : e^{\alpha} = x} \frac{(u-1)(e\overline{e})^{\ell}}{1-(u-1)^2(e\,\overline{e})^{\ell}}\Big)^{-1} \in \mathbf{k}[[u]] \,.$$

(A combinatorial interpretation of these K_x will be given in Section 6.) Let ℓ be a complete $\mathbf{k}$-labelling of $\mathcal{X}$, and define a new labelling $\ell' \colon E(\mathcal{X}) \to \mathbf{k}[[u]]$ by

$$e^{\ell'} = \frac{1}{1-(u-1)^2(e\,\overline{e})^{\ell}} e^{\ell} K_{e^{\omega}} \,.$$

Then our main result is the following:

THEOREM 2.4. *With the definitions of ℓ' and K_x given above, ℓ' is a complete labelling and we have in* $\mathbf{k}[[u]]$ *the equality*

$$\mathfrak{F}(\ell) = K_{\star} \cdot \mathfrak{G}(\ell') \,. \tag{2.1}$$

We now explicit the definitions and main result for the length labelling on a locally finite graph.

DEFINITION 2.5 (Path Series). The integer-valued series

$$G(t) = \sum_{\pi \in [\star,\dagger]} t^{|\pi|} \in \mathbf{N}[[t]]$$

is called the *path series* of $(\mathcal{X}, \star, \dagger)$. The series

$$F(u,t) = \sum_{\pi \in [\star,\dagger]} u^{\mathrm{bc}(\pi)} t^{|\pi|} \in \mathbf{N}[u][[t]] \subset \mathbf{N}[[u,t]]$$

is called the *enriched path series* of $(\mathcal{X}, \star, \dagger)$. Its specialization $F(0,t)$ is called the *proper path series* of $(\mathcal{X}, \star, \dagger)$.

In case $\star = \dagger$, we will call G the *circuit series* of $(\mathcal{X}, \star)$ and F the *enriched circuit series* of $(\mathcal{X}, \star)$. The circuit series is often called the *Green function* of the graph $\mathcal{X}$.

Note that $F(u,t)$ lies in $\mathbf{N}[u][[t]]$ because the number of bumps on a path is smaller than its length, so all monomials in the sum have a u-degree smaller than their t-degree; hence for any fixed t-degree there are only finitely many monomials with same t-degree, because $\mathcal{X}$ is locally finite.

Expressed in terms of length labellings, our main theorem then gives the following result:

COROLLARY 2.6. *Suppose $\mathcal{X}$ is a d-regular graph. Then we have*

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}. \tag{2.2}$$

Proof. Because $\mathcal{X}$ is regular, the K_x defined above do not depend on x and are all equal to

$$K = \left(1 - d\frac{(u-1)t^2}{1-(u-1)^2t^2}\right)^{-1} = \frac{1-(1-u)^2t^2}{1+(d-1+u)(1-u)t^2};$$

thus Theorem 2.4 reads

$$\mathfrak{F}(e \mapsto t) = K \cdot \mathfrak{G}\left(e \mapsto \frac{tK}{1-(1-u)^2t^2} =: \circledast\right).$$

Now writing $\mathfrak{F}(e \mapsto t) = F(u,t)$ and $\mathfrak{G}(e \mapsto \circledast) = G(\circledast)$ completes the proof. $\square$

The special case $u = 1$ of this formula appears as an exercise in [God93, page 72].

The meaning of the corollary is that, for regular graphs, the richer two-variable generating series $F(u,t)$ can be recovered from the simpler $G(t)$. Conversely, G can be recovered from F, for instance because $G(t) = F(1,t)$. Remember it is valid to substitute 1 for u in F, because for any fixed t-degree only finitely many monomials with that t-degree occur in F. In fact, much more is true, as we have the equality

$$G(z) = \frac{1 + \frac{\left(1-\sqrt{1-4(1-u)(d-1+u)z^2}\right)^2}{4(1-u)(d-1+u)z^2}}{1 - \frac{\left(1-\sqrt{1-4(1-u)(d-1+u)z^2}\right)^2}{4(d-1+u)^2z^2}} F\left(u, \frac{1-\sqrt{1-4(1-u)(d-1+u)z^2}}{2(1-u)(d-1+u)z}\right),$$

or after simplification

$$G(z) = \frac{2(d-1+u)}{d-2+u+(d+u)\sqrt{1-4(1-u)(d-1+u)z^2}} \times F\left(u, \frac{1-\sqrt{1-4(1-u)(d-1+u)z^2}}{2(1-u)(d-1+u)z}\right) \tag{2.3}$$

where both sides are to be understood as power series in $\mathbf{N}[[u,z]]$ that actually reside in $\mathbf{N}[[z]]$. Then for any value (say, in $\mathbf{C}$) of u we obtain an expression of G in terms of $F(u,-)$. Of particular interest is the case $u=0$, where (2.3) specializes to

$$G(z)=\frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)z^2}}F\left(0,\frac{1-\sqrt{1-4(d-1)z^2}}{2(d-1)z}\right). \tag{2.4}$$

This equation appears in a slightly different form in [Gri78a].

Similarly, we have in $\mathbf{N}[[t,z,z^{-1}]]$ the equality

$$G(z)=\frac{2}{2-d^2tz+dz\sqrt{d^2t^2+4-4t/z}} \times F\left(1-\frac{dt-\sqrt{d^2t^2+4-4t/z}}{2t},t\right). \tag{2.5}$$

Beware though that (2.5) holds for formal variables z and t; if we were to substitute a real number for t, then the resulting series $G(z)$ would converge absolutely for $\frac{t}{1+(d-1)t^2}\leq z\leq t<\rho$, where ρ is the radius of convergence of $F(1,-)=G$, and in particular not in a neighbourhood of 0.

The equalities (2.3) and (2.5) are easily derived from (2.2) by setting $z=\frac{t}{1+u(d-u)t^2}$ and solving for t and u.

COROLLARY 2.7. *In the setting described above:*

- *If $\mathcal{X}$ is finite, then both F and G are rational series.*
- *If G is rational, then F is rational too.*
- *F is algebraic if and only if G is algebraic.*

The proofs are immediate and follow from the explicit form of (2.2). The converse of the first statement of the preceding corollary will be proved in Section 3.2. The last statement appears in [Gri78a] and [GH97].

In the following section we draw some applications to other fields: group theory and language theory. We give applications of Theorem 2.4 and Corollary 2.6 to some examples of graphs in Section 7, and a derivation of a "cogrowth formula" (as that of Subsection 3.1) for a non-free presentation in Section 8.

We give two proofs of the main result in Sections 4 and 6. The first one, shorter, uses a reduction to finite graphs and their adjacency matrices. The second one is combinatorial and uses the inclusion-exclusion principle. Using the first proof, we obtain in Section 5 an extension of a result by Yasutaka Ihara.

Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

3. APPLICATIONS TO OTHER FIELDS

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

3.1 APPLICATIONS TO RANDOM WALKS ON GROUPS

In this section we show how G is related to random walks and F to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces Π/Ξ, where Ξ does not have to be normal and Π is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have $F(t) = F(0,t)$. We recall the notion of growth of groups:

DEFINITION 3.1. Let Γ be a group generated by a finite symmetric set S. For a $\gamma \in \Gamma$ define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\} .$$

The *growth series* of (Γ, S) is the formal power series

$$f_{(\Gamma,S)}(t) = \sum_{\gamma\in\Gamma} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f_{(\Gamma,S)}(t) = \sum f_n t^n$, the *growth* of (Γ, S) is

$$\alpha(\Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than $|S| - 1$).

Let R be a subset of Γ. The *growth series* of R relative to (Γ, S) is the formal power series

$$f^R_{(\Gamma,S)}(t) = \sum_{\gamma\in R} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f^R_{(\Gamma,S)}(t) = \sum f_n t^n$, define the *growth* of R relative to (Γ, S) as

$$\alpha(R; \Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n} .$$

If X is a transitive right Γ-set, the *simple random walk* on (X,S) is the random walk of a point on X, having probability $1/|S|$ of moving from its current position x to a neighbour $x\cdot s$, for all $s\in S$. Fix a point $\star\in X$, and let p_n be the probability that a walk starting at $\star$ finish at $\star$ after n moves. We define the *spectral radius* (which does not depend on the choice of $\star$) of the random walk as

$$\nu(X,S)=\limsup_{n\to\infty}\sqrt[n]{p_n}\ .$$

A group Π is *quasi-free* if it is a free product of cyclic groups of order 2 and ∞. Equivalently, there exists a finite set S and an involution $\bar{\cdot}\colon S\to S$ such that, as a monoid,

$$\Pi=\langle S\mid s\bar{s}=1\quad\forall s\in S\rangle\ .$$

Π is then said to be *quasi-free on* S. All quasi-free groups on S have the same Cayley graph, which is a regular tree of degree $|S|$.

Every group Γ generated by a symmetric set S is a quotient of a quasi-free group in the following way: let $\bar{\cdot}$ be an involution on S such that for all $s\in S$ we have the equality $\bar{s}=s^{-1}$ in Γ. Then Γ is a quotient of the quasi-free group $\langle S\mid s\bar{s}=1\quad\forall s\in S\rangle$.

The *cogrowth series* (respectively *cogrowth*) of (Γ,S) is defined as the growth series (respectively growth) of $\ker(\pi\colon\Pi\to\Gamma)$ relative to (Π,S), where Π is a quasi-free group on S.

Associated with a group Π generated by a set S and a subgroup Ξ of Π, there is a $|S|$-regular graph $\mathcal{X}$ on which Π acts, called the *Schreier graph* of (Π,S) relative to Ξ. It is given by $\mathcal{X}=(V,E)$, with

$$V=\Xi\backslash\Pi$$

and

$$E=V\times S\,,\quad (v,s)^\alpha=v\,,\quad (v,s)^\omega=vs\,,\quad \overline{(v,s)}=(vs,s^{-1})\,;$$

i.e. two cosets A,B are joined by at least one edge if and only if $AS\supset B$. (This is the Cayley graph of (Π,S) if $\Xi=1$.) There is a circuit in $\mathcal{X}$ at every vertex $\Xi v\in\Xi\backslash\Pi$ such that $s\in v^{-1}\Xi v$ for some $s\in S$; and there is a multiple edge from Ξv to Ξw in $\mathcal{X}$ if there are $s,t\in v^{-1}\Xi w$ with $s\neq t\in S$.

COROLLARY 3.2 (of Corollary 2.6). *Let Π be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

Let $\Xi < \Pi$ be a subgroup of Π. Let $\nu = \nu(\Xi \backslash \Pi, S)$ denote the spectral radius of the simple random walk on $\Xi \backslash \Pi$ generated by S; and $\alpha = \alpha(\Xi; \Pi, S)$ denote the relative growth of Ξ in Π. Then we have

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \leq \sqrt{|S|-1}. \end{cases} \tag{3.1}$$

Proof. Let $\mathcal{X}$ be the Schreier graph of (Π, S) relative to Ξ defined above. Fix the endpoints $\star = \dagger = \Xi$, the coset of 1, and give $\mathcal{X}$ the length labelling. Let G and F be the circuit and proper circuit series of $\mathcal{X}$. In this setting, expressing $F(t) = \sum f_n t^n$ and $G(t) = \sum g_n t^n$, we see that $|S|\nu$ is the growth rate $\limsup \sqrt[n]{g_n}$ of circuits in $\mathcal{X}$, and α the growth rate $\limsup \sqrt[n]{f_n}$ of proper circuits in $\mathcal{X}$. As both F and G are power series with non-negative coefficients, $1/(|S|\nu)$ is the radius of convergence of G and $1/\alpha$ the radius of convergence of F. Let $d = |S|$ and consider the function

$$(t)\phi = \frac{t}{1+(d-1)t^2} .$$

This function is strictly increasing for $0 \leq t < 1/\sqrt{d-1}$, has a maximum at $t = 1/\sqrt{d-1}$ with $(t)\phi = 1/(2\sqrt{d-1})$, and is strictly decreasing for $t > 1/\sqrt{d-1}$.

First we suppose that $\alpha \geq \sqrt{d-1}$, so ϕ is monotonously increasing on $[0, 1/\alpha]$. We set $u = 1$ in (2.2) and note that, for $t < 1$, it says that F has a singularity at t if and only if G has a singularity at $(t)\phi$. Now as $1/\alpha$ is the singularity of F closest to 0, we conclude by monotonicity of ϕ that the singularity of G closest to 0 is at $(1/\alpha)\phi$; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1+(d-1)/\alpha^2} = (1/\alpha)\phi .$$

Suppose now that $\alpha < \sqrt{d-1}$. If $d\nu < 2\sqrt{d-1}$, the right-hand side of (2.2) would be bounded for all $t \in \mathbf{R}$ while the left-hand side diverges at $t = 1$. If $d\nu > 2\sqrt{d-1}$, there would be a $t \in [0, 1/\sqrt{d-1}[$ with $(t)\phi = d\nu$; and F would have a singularity at $t < 1/\alpha$. The only case left is $d\nu = 2\sqrt{d-1}$. $\square$

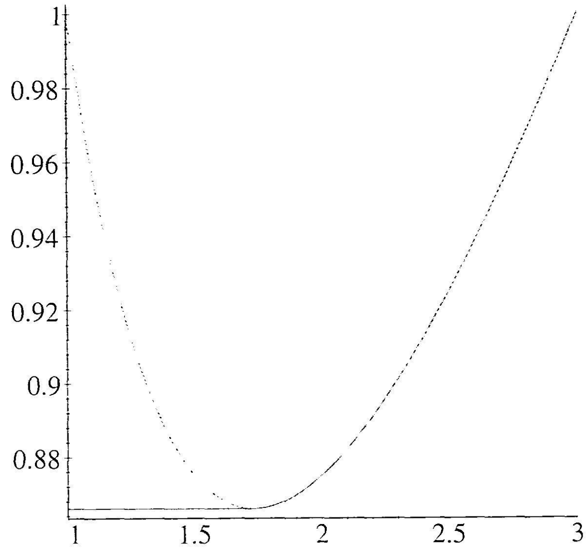

FIGURE 1

The function $\alpha \mapsto \nu$ relating cogrowth and spectral radius (for $d = 4$)

COROLLARY 3.3 (Grigorchuk [Gri78b]). *Let Γ be a group generated by a symmetric finite set S, let ν denote the spectral radius of the simple random walk on Γ, and let α denote the cogrowth of (Γ, S). Then*

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else}. \end{cases} \tag{3.2}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

Proof. Present Γ as Π/Ξ, with Π a quasi-free group and Ξ the normal subgroup of Π generated by the relators in Γ, and apply Corollary 3.2. $\square$

We note in passing that if $\alpha < \sqrt{|S|-1}$, then necessarily $\alpha = 0$. Equivalently, we will show that if $\alpha < \sqrt{|S|-1}$, then $\Xi = 1$, so the Cayley graph $\mathcal{X}$ is a tree. Indeed, suppose $\mathcal{X}$ is not a tree, so it contains a circuit λ at $\star$. As $\mathcal{X}$ is transitive, there is a translate of λ at every vertex, which we will still write λ. There are at least $|S|(|S|-1)^{t-2}(|S|-2)$ paths p of length t in $\mathcal{X}$ starting at $\star$ such that the circuit $p\lambda\overline{p}$ is proper; thus

$$\alpha \geq \limsup_{t\to\infty} \sqrt[2t+|\lambda|]{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}\,.$$

In fact it is known that $\alpha > \sqrt{|S|-1}$; see [Pas93].

3.2 The series F and G on their circle of convergence

In this subsection we study the singularities the series F and G may have on their circle of convergence. The smallest positive real singularity has a special importance:

Definition 3.4. For a series $f(t)$ with positive coefficients, let $\rho(f)$ denote its radius of convergence. Then f is *$\rho(f)$-recurrent* if

$$\lim_{t\to\rho(f)} f(t) = \infty .$$

Otherwise, it is *$\rho(f)$-transient.*

As typical examples, $1/(\rho - t)$ is ρ-recurrent, as are all rational series; $\sqrt{\rho - t}$ is ρ-transient, while $1/\sqrt{\rho - t}$ is not.

To study the singularities of F or G, we may suppose that $\star = \dagger$; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of F and G do not depend on the choice of $\star$ and $\dagger$. We make that assumption for the remainder of the subsection. We will also suppose throughout that $\mathcal{X}$ is d-regular, that the radius of convergence of F is $1/\alpha$ and the radius of convergence of G is $1/(d\nu) = 1/\beta$.

Definition 3.5. Let $\mathcal{X}$ be a connected graph. A *proper cycle* in $\mathcal{X}$ is a proper circuit $(\pi_1, \dots, \pi_n)$ such that $\overline{\pi_1} \neq \pi_n$. The *proper period p* and *strong proper period p_s* are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\},$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists}$$
$$\text{a proper cycle } \pi \text{ in } \mathcal{B}(x,n) \text{ with } |\pi| = n\},$$

where by convention the gcd of the empty set is ∞. The graph $\mathcal{X}$ is *strongly properly periodic* if $p = p_s$.

The *period q* and *strong period q_s* of $\mathcal{X}$ are defined analogously with 'proper cycle' replaced by 'circuit'. $\mathcal{X}$ is *strongly periodic* if $q = q_s$.

Theorem 3.6 (Cartwright [Car92]). *Let $\mathcal{X}$ have proper period p and strong proper period p_s. Then the singularities of F on its circle of convergence are among the*

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s .$$

If moreover $\mathcal{X}$ is strongly properly periodic, the singularities of F on its circle of convergence are precisely these numbers.

Let $\mathcal{X}$ have period q and strong period q_s. Then the singularities of G on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \ldots, q_s .$$

If moreover $\mathcal{X}$ is strongly periodic, the singularities of G on its circle of convergence are precisely these numbers.

If $\mathcal{X}$ is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant N such that for all $x \in V(\mathcal{X})$ there is at x a circuit of odd length bounded by N, then $\mathcal{X}$ is strongly 1-periodic; otherwise $\mathcal{X}$ is strongly 2-periodic. The singularities of G on its circle of convergence are then at $1/\beta$, and also at $-1/\beta$ if $\mathcal{X}$ is strongly periodic with period 2.

If $\mathcal{X}$ is not strongly periodic, there may be one or two singularities on G's circle of convergence; consider for instance the 4-regular tree, and at a vertex $\star$ delete two or three edges replacing them by loops. The resulting graphs $\mathcal{X}_2$ and $\mathcal{X}_3$ are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$\begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned} \tag{3.3}$$

G_2 has singularities at $\pm 1/\sqrt{12}$ on its circle of convergence, while G_3 has only $2/7$ as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if $\beta < d$ the singularities of F on its circle of convergence are in bijection with those of G, so are at $1/\alpha$ and possibly $-1/\alpha$, if $\mathcal{X}$ is strongly two-periodic. If $\beta = d$, though, $\mathcal{X}$ can have any strong proper period; consider for example the cycles on length k studied in Section 7.2: they are strongly properly k-periodic.

The forthcoming simple result shows how $\mathcal{X}$ can be approximated by finite subgraphs.

LEMMA 3.7. *Let $\mathcal{X}$ be a graph and x, y two vertices in $\mathcal{X}$. Let $\mathfrak{G}_{x,y}$ and $\mathfrak{F}_{x,y}$ be the path series and enriched path series respectively from x to y in $\mathcal{X}$, and let $\mathfrak{G}^n_{x,y}$ and $\mathfrak{F}^n_{x,y}$ be the path series and enriched path series respectively from x to y in the ball $\mathcal{B}(x,n)$ (they are 0 if $y \notin \mathcal{B}(x,n)$). Then*

$$\lim_{n\to\infty} \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}, \qquad \lim_{n\to\infty} \mathfrak{F}^n_{x,y} = \mathfrak{F}_{x,y}.$$

Proof. Recall that $\lim \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}$ means that both terms are sums of paths, say A_n and A, such that the minimal length of paths in the symmetric difference $A_n \triangle A$ tends to infinity. Now the difference between $\mathfrak{G}^n_{x,y}$ and $\mathfrak{G}_{x,y}$ consists only of paths in $\mathcal{X}$ that exit $\mathcal{B}(x,n)$, and thus have length at least $2n - \delta(x,y) \to \infty$. The same argument holds for $\mathfrak{F}$. $\square$

DEFINITION 3.8. The graph $\mathcal{X}$ is *quasi-transitive* if $\mathrm{Aut}(\mathcal{X})$ acts with finitely many orbits.

LEMMA 3.9. *Let $\mathcal{X}$ be a regular quasi-transitive connected graph with distinguished vertex $\star$, and let f_n and g_n denote respectively the number of proper circuits and circuits at $\star$ of length n. Then*

$$\limsup_{n\to\infty} g_n/\beta^n = \limsup_{n\to\infty} f_n/\alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

Proof. If $\mathcal{X}$ is finite, then $\beta = d$, the degree of $\mathcal{X}$; after a large even number of steps, a random walk starting at $\star$ will be uniformly distributed over $\mathcal{X}$ (or over the vertices at even distance of $\star$, in case all circuits have even length). A long enough walk then has probability $1/|\mathcal{X}|$ (or $2/|\mathcal{X}|$ if all circuits have even length) of being a circuit.

If $\mathcal{X}$ is infinite, we consider two cases. If $G(1/\beta) < \infty$, i.e. G is $1/\beta$-transient, the general term g_n/β^n of the series $G(1/\beta)$ tends to 0. If G is $1/\beta$-recurrent, then, as $\mathcal{X}$ is quasi-transitive, $\beta = d$ by [Woe98, Theorem 7.7]. We then approximate $\mathcal{X}$ by the sequence of its balls of radius R, by Lemma 3.7:

$$\lim_{n\to\infty} \frac{g_n}{\beta^n} = \lim_{R,n\to\infty} \frac{g_{R,n}}{d^n} = \lim_{R\to\infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star,R)|} = 0,$$

where we expand the circuit series of $\mathcal{B}(\star,R)$ as $\sum g_{R,n}t^n$.

The same proof holds for the f_n. Its particular case where $\mathcal{X}$ is a Cayley graph appears in [Woe83]. $\square$

Note that if $\mathcal{X}$ is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if $\mathcal{X}$ is transient or null-recurrent then the common limsup is 0. If $\mathcal{X}$ is positive-recurrent then the limsups are normalized coefficients of $\mathcal{X}$'s Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary d-regular graphs: consider for instance the graph $\mathcal{X}_3$ described above. Its circuit series G_3, given in (3.3), has radius of convergence $1/\beta = 2/7$, and one easily checks that all its coefficients g_n satisfy $g_n/\beta^n \geq 1/2$.

We obtain the following characterization of rational series:

THEOREM 3.10. *For regular quasi-transitive connected graphs $\mathcal{X}$, the following are equivalent:*

1. *$\mathcal{X}$ is finite;*
2. *$G(t)$ is a rational function of t;*
3. *$F(t)$ is a rational function of t, and $\mathcal{X}$ is not an infinite tree.*

Proof. By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case $F(t) = 1$, Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that $F(t) = \sum f_n t^n$ is rational, not equal to 1. As the f_n are positive, F has a pole, of multiplicity m, at $1/\alpha$. There is then a constant $a > 0$ such that $f_n > a\binom{n}{m-1}\alpha^n$ for infinitely many values of n [GKP94, page 341]. It follows by Lemma 3.9 that $m = 1$ and the graph $\mathcal{X}$ is finite, of cardinality at most $1/a$. □

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

3.3 APPLICATION TO LANGUAGES

Let S be a finite set of cardinality d and let $\bar{\cdot}$ be an involution on S. A *word* is an element w of the free monoid S^*. A *language* is a set L of words. The language L is called *saturated* if for any $u, v \in S^*$ and $s \in S$ we have

$$uv \in L \Longleftrightarrow us\bar{s}v \in L;$$

that is to say, L is stable under insertion and deletion of subwords of the form $s\bar{s}$. The language L is called *desiccated* if no word in L contains a subword of the form $s\bar{s}$. Given a language L we may naturally construct its *saturation*

$\langle L \rangle$, the smallest saturated language containing L, and its *desiccation* $\widehat{L}$, the largest desiccated language contained in L.

Let Σ be the monoid defined by generators S and relations $s\bar{s} = 1$ for all $s \in S$:

$$\Sigma = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle . \tag{3.4}$$

This is a free product of free groups and order-two groups; if $\bar{\cdot}$ is fixed-point-free, Σ is a free group. Write ϕ for the canonical projection from S^* to Σ. Let $\mathbf{k} = \mathbf{Z}[\Sigma]$ be its monoid ring. Then given a language $L \subset S^*$ we may define its *growth series* $\Theta(L)$ as

$$\Theta(L) = \sum_{w \in L} w^\phi t^{|w|} \in \mathbf{k}[[t]] .$$

This notion of growth series with coefficients was introduced by Fabrice Liardet in his doctoral thesis [Lia96], where he studied *complete growth functions* of groups.

THEOREM 3.11. *For any language L there holds*

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} , \tag{3.5}$$

where $d = |S|$.

Proof. For any language there exists a unique minimal (possibly infinite) automaton recognising it ([Eil74, § III.5] is a good reference). Let $\mathcal{X}$ be the minimal automaton recognising $\langle L \rangle$. Recall that this is a graph with an initial vertex $\star$, a set of terminal vertices T and a labelling $\ell' \colon E(\mathcal{X}) \to S$ of the graph's edges such that the number of paths labelled w, starting at $\star$ and ending at a $\tau \in T$ is 1 if $w \in L$ and 0 otherwise. Extend the labelling ℓ' to a labelling $\ell \colon E(\mathcal{X}) \to \mathbf{k}[[t]]$ by

$$e^\ell = t \cdot (e^{\ell'})^\phi .$$

Because $\langle L \rangle$ is saturated, and $\mathcal{X}$ is minimal, $(\bar{e})^\ell = \overline{e^\ell}$; then $\widehat{L}$ is the set of labels on proper paths from $\star$ to some $\tau \in T$. Choosing in turn all $\tau \in T$ as $\dagger$, we obtain growth series F_τ, G_τ counting the formal sum of paths and proper paths from $\star$ to τ. It then suffices to write

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\sum_{\tau \in T} F_\tau(t)}{1-t^2} = \frac{\sum_{\tau \in T} G_\tau\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} . \qquad \square$$

The following result is well-known:

THEOREM 3.12 (Müller & Schupp [MS81, MS83]). *Let Γ be a finitely generated group, presented as a quotient Σ/Ξ with Σ as in (3.4). Then $\Theta(\Xi)$ is an algebraic series (i.e. satisfies a polynomial equation over $\mathbf{k}[t]$) if and only if Σ/Ξ is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to $\mathbf{k}$-matrices and $\mathbf{k}[[u]]$-matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices $x, y \in V(\mathcal{X})$ let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi\in[x,y]} \pi^\ell, \qquad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi\in[x,y]} u^{\mathrm{bc}(\pi)}\pi^\ell$$

be the path and enriched path series from x to y; for ease of notation we will leave out the labelling ℓ if it is obvious from the context. Let $\delta_{x,y}$ denote the Kronecker delta, equal to 1 if $x = y$ and 0 otherwise. For any $v \in \mathbf{k}$, let $[v]_x^y$ denote the $V(\mathcal{X}) \times V(\mathcal{X})$ matrix with zeros everywhere except at (x, y), where it has value v. Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e\in E(\mathcal{X}):\, e^\alpha = x} e^\ell \mathfrak{G}_{e^\omega,y}$$

so that if

$$A = \sum_{e\in E(\mathcal{X})} [e^\ell]_{e^\alpha}^{e^\omega}$$

be the adjacency matrix of $\mathcal{X}$, with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y\in V(\mathcal{X})} = \frac{1}{1-A},$$

an equation holding between $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{k}$.

Similarly, letting $\mathfrak{F}_{x,e,y}$ count the paths from x to y that start with the edge e,

$$\mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^\alpha = x} \mathfrak{F}_{x,e,y} ,$$

$$\mathfrak{F}_{x,e,y} = e^\ell \left(\mathfrak{F}_{e^\omega,y} + (u-1)\mathfrak{F}_{e^\omega,\overline{e},y} \right) ,$$

$$\mathfrak{F}_{e^\omega,\overline{e},y} = \overline{e}^\ell \left(\mathfrak{F}_{x,y} + (u-1)\mathfrak{F}_{x,e,y} \right) ;$$

these last two lines solve to

$$\mathfrak{F}_{x,e,y} = \left(1 - (u-1)^2 (e\overline{e})^\ell \right)^{-1} \left(e^\ell \mathfrak{F}_{e^\omega,y} + (u-1)(e\overline{e})^\ell \mathfrak{F}_{x,y} \right) ,$$

which we insert in the first line to obtain

$$K_x^{-1} \mathfrak{F}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}): e^\alpha = x} \frac{e^\ell}{1 - (u-1)^2 (e\,\overline{e})^\ell} K_{e^\omega} \cdot K_{e^\omega}^{-1} \mathfrak{F}_{e^\omega,y} .$$

Thus if we let

$$e^{\ell'} = \frac{e^\ell}{1 - (u-1)^2 (e\,\overline{e})^\ell} K_{e^\omega} , \qquad A' = \sum_{e \in E(\mathcal{X})} [e^{\ell'}]_{e^\alpha}^{e^\omega} , \tag{4.1}$$

we obtain

$$(K_x^{-1} \mathfrak{F}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A'} \tag{4.2}$$

and the proof is finished in the case that $\mathcal{X}$ is finite, because the matrix A' is precisely that obtained from A by substituting ℓ' for ℓ.

If $\mathcal{X}$ has infinitely many vertices, we approximate it, using Lemma 3.7, by finite graphs. Denote by $\mathfrak{F}^n_{\star,\dagger}(\ell)$ and $\mathfrak{G}^n_{\star,\dagger}(\ell')$ the enriched path series and path series respectively in $\mathcal{B}(\star, n)$, and write

$$K_\star \cdot \mathfrak{F}(\ell) = \lim_{n \to \infty} \mathfrak{F}^n_{\star,\dagger}(\ell) = \lim_{n \to \infty} \mathfrak{G}^n_{\star,\dagger}(\ell') = \mathfrak{G}(\ell')$$

to complete the proof.

5. Graphs and matrices

Graphs can be studied through their *adjacency* and *incidence* matrices. We give here the relevant definitions and obtain an extension of a theorem by Hyman Bass [Bas92] on the Ihara-Selberg zeta function. We will use power series with coefficients in a matrix ring, and fractional expressions in matrices; by convention, we understand 'X/Y' as '$X \cdot Y^{-1}$'.

DEFINITION 5.1. Let $\mathcal{X}$ be a finite graph. The *edge-adjacency* and *inversion* matrices of $\mathcal{X}$, respectively B and J, are $E(\mathcal{X}) \times E(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$B_{e,f} = \begin{cases} 1 & \text{if } e^\omega = f^\alpha \\ 0 & \text{else,} \end{cases} \qquad J_{e,f} = \begin{cases} 1 & \text{if } \bar{e} = f \\ 0 & \text{else.} \end{cases}$$

The *vertex-adjacency* and *degree* matrices of $\mathcal{X}$, respectively A and D, are $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{Z}$ defined by

$$A_{v,w} = \left|\{e \in E(\mathcal{X}) \mid e^\alpha = v \text{ and } e^\omega = w\}\right|, \quad D_{v,w} = \begin{cases} \deg(v) & \text{if } v = w, \\ 0 & \text{else.} \end{cases}$$

A *cycle* is the equivalence class of a circuit under cyclic permutation of its edges. A *proper cycle* is a cycle all of whose representatives are proper circuits. A cycle is *primitive* if none of its representatives can be written as π^k for some $k \geq 2$. The *cyclic bump count* $\mathrm{cbc}(\pi)$ of a circuit $\pi = (\pi_1, \dots, \pi_n)$ is

$$\mathrm{cbc}(\pi) = \left|\{i = 1, \dots, n \mid \pi_i = \overline{\pi_{i+1}}\}\right| ,$$

where the edge π_{n+1} is understood to be π_1.

The matrices given above are related to paths in $\mathcal{X}$ as follows: Consider first the matrix

$$M = \mathbf{1} - (B - (1-u)J)t .$$

Then the (e,f) coefficient of M^{-1} is precisely

$$\sum_{\pi : \pi_1 = e, \pi^\omega = f^\alpha} u^{\mathrm{bc}(\pi f)} t^{|\pi|} .$$

This is clear because the series expansion of M^{-1} is the sum of sequences of $(B-J)t$ (contributing edges with no bump) and Jut (contributing edges with bumps), with an extra factor of u in case the path ends in $\bar{f}$. As a consequence,

LEMMA 5.2. *Let*

$$X_E = \frac{\mathbf{1} + (1-u)Jt - M}{Mt} = \frac{B}{\mathbf{1} - (B - (1-u)J)t} .$$

Then the (e,f) coefficient of X_E counts the non-trivial paths starting with e and ending at f^α, with t-weight shifted one down:

$$(X_E)_{e,f} = \sum_{\pi : \pi_1 = e, \ \pi^\omega = f^\alpha} u^{\mathrm{bc}(\pi)} t^{|\pi| - 1} .$$

Likewise, consider the matrix

$$P = \mathbf{1} - At + (1-u)(D - (1-u)\mathbf{1})t^2 .$$

The following lemma will be a consequence of the computations in the next section.

LEMMA 5.3. *Let*

$$X_V = \frac{(1-(1-u)^2t^2)\mathbf{1} - P}{Pt} = \frac{A - (1-u)Dt}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} .$$

Then the (v,w) coefficient of X_V counts the non-trivial paths starting at v and ending at w, with t-weight shifted one down:

$$(X_V)_{v,w} = \sum_{\pi\,:\,\pi^\alpha = v,\ \pi^\omega = w} u^{\mathrm{bc}(\pi)} t^{|\pi|-1} .$$

Proof. We will show the matrix $\mathbf{1} + X_V t$ has as (v,w) coefficient the enriched path series from v to w. By simple calculation

$$\mathbf{1} + X_V t = \frac{\mathbf{1} - (1-u)^2t^2}{\mathbf{1} - At + (1-u)(D-(1-u)\mathbf{1})t^2} = \frac{K^{-1}}{\mathbf{1} - A'} ,$$

where K and A' are given by

$$K = \frac{\mathbf{1} + (1-u)(D - 1 + u)t^2}{1-(1-u)^2t^2}, \qquad A' = \frac{AKt}{1-(1-u)^2t^2} .$$

K is a diagonal matrix and the coefficient $K_{x,x}$ is precisely K_x for the length labelling, while the matrix A' is the matrix of (4.1) in the previous section. The result then follows from Equation (4.2). □

In particular, the two matrices X_E and X_V have the same trace, as this trace counts all the non-trivial circuits π in $\mathcal{X}$, with weight $u^{\mathrm{bc}(\pi)}t^{|\pi|-1}$.

We now state and prove an extension of a theorem by Bass [Bas92, FZ98, Nor96]:

THEOREM 5.4. *Let C be a set of representatives of primitive cycles in $\mathcal{X}$, and form the zeta function of $\mathcal{X}$*

$$\zeta(u,t) = \prod_{\gamma \in C} \frac{1}{1 - u^{\mathrm{cbc}(\gamma)} t^{|\gamma|}} .$$

(The choice of representatives does not change the zeta function.) Then ζ^{-1} *is a polynomial in* u *and* t *and can be expressed as*

$$\frac{1}{\zeta(u,t)} = \det M \tag{5.1}$$

$$= (1+(1-u)t)^n(1-(1-u)^2t^2)^{m-|V(\mathcal{X})|}\det P\,, \tag{5.2}$$

where

$$n = |\{e \in E(\mathcal{X}) \mid e = \overline{e}\}|\,, \qquad 2m = |\{e \in E(\mathcal{X}) \mid e \neq \overline{e}\}|\,.$$

The special case $u = n = 0$ of this result was stated and proved in the given sources. We will prove the general statement, using a result of Shimson Amitsur:

THEOREM 5.5 (Amitsur [Ami80,RS87]). *Let* $X_1, \dots, X_k$ *be square matrices of the same dimension over an arbitrary ring. Let* S *contain one representative up to cyclic permutation of words over the alphabet* $\{1, \dots, k\}$ *that are primitive, i.e. such that none of their cyclic permutations are proper powers of a word (*S *is infinite as soon as* $k > 1$*). For* $p = i_1 \dots i_n \in S$ *set* $X_p = X_{i_1} \dots X_{i_n}$*. Then*

$$\det(\mathbf{1} - (X_1 + \cdots + X_k)t) = \prod_{p \in S} \det(\mathbf{1} - X_p t^{|p|})\,,$$

considered as an equality of power series in t *over the matrix ring.*

The equality (5.1) then follows; indeed, for all edges $e \in E(\mathcal{X})$ let X_e be the $E(\mathcal{X}) \times E(\mathcal{X})$ matrix whose e-th row is the e-th row of $B - (1-u)J$, and whose other rows are 0. Then clearly $\mathbf{1} - \sum_{e \in E(\mathcal{X})} X_e t = M$ and, for any sequence of edges π,

$$\det(\mathbf{1} - X_\pi t^{|\pi|}) = \begin{cases} 1 - u^{\mathrm{cbc}(\pi)} t^{|\pi|} & \text{if } \pi \text{ is a circuit,} \\ 1 & \text{else,} \end{cases}$$

so equality of $\zeta(u,t)$ and $\det M$ follows from Amitsur's Theorem.

To prove (5.2), we use the following result, whose proof relies on Newton's formulas relating the trace of powers of X and the characteristic polynomial of X:

PROPOSITION 5.6 ([Ami80, Equation 4.4)]. *Let X be a power series in t over a matrix ring, such that $X(0) = \mathbf{1}$. Then*

$$\det X = \exp\Bigl(-\int \operatorname{tr}\Bigl(\frac{\mathbf{1}-X}{Xt}\Bigr)dt\Bigr)\,,$$

where the integration is the formal linear operation on power series that maps t^r to $t^{r+1}/(r+1)$.

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned}\frac{\det M}{(1+(1-u)t)^n(1-(1-u)^2t^2)^m} &= \det\frac{M}{\mathbf{1}+(1-u)Jt}\\ &= \exp\Bigl(-\int \operatorname{tr}\frac{\mathbf{1}+(1-u)Jt-M}{Mt}dt\Bigr)\\ &= \exp\Bigl(-\int \begin{matrix}\text{series counting non-trivial circuits,}\\ \text{length shifted down by one}\end{matrix}\; dt\Bigr)\\ &= \exp\Bigl(-\int \operatorname{tr}\frac{(1-(1-u)^2t^2)\mathbf{1}-P}{Pt}dt\Bigr)\\ &= \det\frac{P}{1-(1-u)^2t^2} = \frac{\det P}{(1-(1-u)^2t^2)^{|V(\mathcal{X})|}}\,.\end{aligned}$$

6. SECOND PROOF OF THEOREM 2.4

Let $P = [\star, \dagger]$ be the set of paths in $\mathcal{X}$ from $\star$ to $\dagger$. As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi = \mathbf{Z}[[P]]$, the $\mathbf{Z}$-module of functions from the set of paths to $\mathbf{Z}$. We embed subsets of P in Π by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A\,,\quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A\,,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathcal{B}$ be the subset of bounded non-negative elements of Π (i.e. functions f such that there is a constant N with $0 \leq (\pi)f < N$ for all paths π). If ℓ is a complete labelling of $\mathcal{X}$, there is an induced labelling $\ell_*\colon \mathcal{B} \to \mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi\in P}(\pi)f\pi^\ell\,.$$

Note that the sum, although infinite, defines an element of $\mathbf{k}$ due to the fact that ℓ is complete.

DEFINITION 6.1 (Bump Scheme). Let $e \in E(\mathcal{X})$ and $v \in V(\mathcal{X})$. A *squiggle along* e is a sequence $(e, \overline{e}, \dots, e, \overline{e})$. A *squiggle at* v is a squiggle along e for some edge e such that $e^\alpha = v$.

Let $\pi = (v_0, e_1, \dots, e_n, v_n)$ be a path of length n in $\mathcal{X}$. A *bump scheme* for π is a pair $B = \big((\beta_0, \dots, \beta_n), (\gamma_1, \dots, \gamma_n)\big)$, with

- for all $i \in \{0, \dots, n\}$, a finite (possibly empty) sequence $\beta_i = (\beta_{i,1}, \dots, \beta_{i,t_i})$ of squiggles at v_i;
- for all $i \in \{1, \dots, n\}$, a squiggle γ_i along e_i.

The *weight* $|B|$ of the bump scheme B is defined as

$$|B| = \sum_{i=0}^{n} \sum_{j=1}^{t_i} (|\beta_{i,j}| - 1) + \sum_{i=1}^{n} |\gamma_i| .$$

Given a path π and a bump scheme $B = (\beta, \gamma)$ for π, we obtain a new path $\pi \vee B \in P$, by setting

$$\pi \vee B = \beta_{0,1} \cdot \dots \cdot \beta_{0,t_0} \gamma_1 e_1 \beta_{1,1} \cdot \dots \cdot \gamma_n e_n \beta_{n,1} \cdot \dots \cdot \beta_{n,t_n} ,$$

where the product denotes concatenation.

We now define a linear map $\phi \colon \Pi \to \Pi[[u]]$ by setting, for $f \in \Pi$ and $\pi \in P$,

$$(\pi)\big((f)\phi\big) = \sum_{(\rho,B)\,:\,\rho \vee B = \pi} (u-1)^{|B|} (\rho) f ,$$

where the sum ranges over all pairs (ρ, B) where $\rho \in P$ and B is a bump scheme for ρ such that $\rho \vee B = \pi$. Note that the sum is finite because the edges of ρ and of B form subsets of those of π.

LEMMA 6.2. *For any path* π *we have*

$$(\pi)\big((\chi_P)\phi\big) = u^{\mathrm{bc}(\pi)} . \tag{6.1}$$

Proof. Say $\pi = (\pi_1, \dots, \pi_n)$ has $m \geq 0$ bumps, at indices $b_1, \dots, b_m$ so that $\pi_{b_i} = \overline{\pi_{b_{i+1}}}$. We will show that the evaluation at π of the left-hand side of (6.1) yields u^m.

We claim there is a bijection between the subsets C of $\{1, \dots, m\}$ and the pairs (ρ_C, B_C) where ρ_C is a path and B_C is a bump scheme for ρ_C with $\pi = \rho_C \vee B_C$; and further $|B_C| = |C|$.

First, take a ρ and a $B = (\beta, \gamma)$ such that $\rho \vee B = \pi$. The path $\rho \vee B$ is obtained by shuffling together the edges of ρ and B, and this partitions the edges of π in two classes, namely *(i)* those coming from ρ and *(ii)* those coming from β and γ. Let $C \subset \{1,\dots,m\}$ be the indices of the bumps b_i in π coming from B, i.e. such that π_{b_i} and $\pi_{b_{i+1}}$ belong to the class *(ii)*. One direction of the bijection is then $(\rho, B) \mapsto C$.

Conversely, given a subset C consider the set $D = \{b_i \mid i \in C\}$. Split it in maximal-length runs of consecutive integers $D = D_1 \sqcup \cdots \sqcup D_t$. For all runs D_i do the following: to $D_i = \{j, j+1, \dots, j+2k-1\}$ of even cardinality associate a squiggle γ_j of length $2k$ along π_j; to $D_i = \{j, j+1, \dots, j+2k-2\}$ of odd cardinality associate a squiggle $\beta_{j,l}$ of length $2k$ at v_{j-1}; then delete in π the edges $\pi_j, \dots, \pi_{j+2k-1}$. This process constructs a bump scheme $B = (\beta, \gamma)$ while pruning edges of π, giving a path γ with $\gamma \vee B = \pi$. These two constructions are inverses, proving the claimed bijection.

It now follows that

$$(\pi)(\chi_P)\phi = \sum_{C \in \{1,\dots,m\}} (u-1)^{|B_C|} = \sum_{r=0}^{m} (u-1)^r \binom{m}{r} = u^m . \qquad \square$$

Let $\ell' \colon E(\mathcal{X}) \to \mathbf{k}[[u]]$ be defined by

$$e^{\ell'} = \frac{1}{1-(e\,\overline{e})^{\ell}(1-u)^2} e^{\ell} K_{e^{\omega}} .$$

We prove Theorem 2.4 by noting that $\mathfrak{G}(\ell) = (\chi_P)\ell_*$, that $\mathfrak{F}(\ell) = (\chi_P\phi)\ell_*$, and that for any $f \in \Pi$ we have $(f\phi)\ell_* = K_\star(f)\ell'_*$. To prove this last equality, take a path $\pi = (\pi_1, \dots, \pi_n)$ on vertices $v_0, \dots, v_n$. Then

$$(\chi_{\{\pi\}}\phi)\ell_* = \sum_B (u-1)^{|B|}(\pi \vee B)^{\ell} ,$$

where the sum ranges over all bump schemes for π, and

$$K_\star \pi^{\ell'} = K_{v_0} \frac{1}{1-(u-1)^2(\pi_1\overline{\pi_1})^{\ell}} \pi_1^{\ell} K_{v_1} \cdot \ldots \cdot \frac{1}{1-(u-1)^2(\pi_n\overline{\pi_n})^{\ell}} \pi_n^{\ell} K_{v_n} .$$

It is clear these last two lines are equal; for the power series expansion of the K_{v_i} correspond to all the possible squiggle sequences β_i at v_i, and the power series expansion of the $1/(1-(u-1)^2(\pi_i\overline{\pi_i})^{\ell})$ correspond to all possible squiggles γ_i along π_i.

7. EXAMPLES

We give here examples of regular graphs and when possible compute independently the series F and G. In some cases it will be easier to compute F, while in others it will be simpler to compute G first. In all cases, once one of F and G has been computed, the other one can be obtained using Corollary 2.6.

In all the examples the graphs are vertex transitive, so the choice of $\star$ is unimportant. To simplify the computations we choose $\dagger = \star$ and the length labelling.

7.1 COMPLETE GRAPHS

Let $\mathcal{X} = K_v$, the complete graph on $v \geq 3$ vertices. Its degree is $d = v-1$. To compute F and G, choose three distinct vertices $\star, \$, \#$ (the choice is unimportant as K_v is three-transitive). Define growth series

$F(u,t)$ the growth series of circuits based at $\star$;
$F'(u,t)$ the growth series of paths π from $\$$ to $\star$ with $\pi_1^\omega = \#$;
$F''(u,t)$ the growth series of paths π from $\$$ to $\star$ with $\pi_1^\omega = \star$.

Then

$$\begin{aligned} F &= 1 + (v-1)t\Big[(v-2)F' + uF''\Big], \\ F' &= t\Big[F'' + (v-3+u)F'\Big], \\ F'' &= t\Big[1 + (F-1)\frac{v-2+u}{v-1}\Big]. \end{aligned}$$

Indeed the first line states that a circuit at $\star$ is either the trivial circuit at $\star$, or a choice of one of $v-1$ edges to another point (call it $\$$), followed by a path from $\$$ to $\star$; this path can first go to any vertex of the $v-2$ vertices (say ($\#$)) different from $\star$ and $\$$, and thus contribute F', or go back to $\star$ and contribute F'' and a bump.

The second equation says that a path from $\$$ to $\star$ starting by going to $\#$ can either continue to $\star$, contributing F', go to any of the $v-3$ other vertices contributing F', or come back to $\$$, contributing F' and a bump.

The third line says that a path from $\$$ to $\star$ starting by going to $\star$ continues as a circuit at $\star$; but if the circuit is non-trivial, then one out of $v-1$ times a bump will be contributed.

Solving the system, we obtain

$$F(u,t)=\frac{1+(1-u)t}{1-(v-2+u)t}\cdot\frac{1-(v-2)t+(1-u)(v-2+u)t^2}{1+t+(1-u)(v-2+u)t^2}.$$

We then compute

$$G(t)=F(1,t)=\frac{1-(v-2)t}{(1+t)(1-(v-1)t)},$$
$$F(0,t)=\frac{(1+t)(1-(v-2)t+(v-2)t^2)}{(1-(v-2)t)(1+t+(v-2)t^2)}.$$

7.2 Cycles

Let $\mathcal{X}=C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0,t)=\frac{1+t^k}{1-t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n=\sum_{i\in\mathbf{Z}:\,i\equiv 0\ [k],\ i\equiv n\ [2]}\binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t)=\frac{1}{k}\sum_{\zeta^k=1}\frac{1}{1-\left(\zeta+\zeta^{-1}\right)t}=\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{1-2\cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t)=\frac{(2t)^2+\left(1-\sqrt{1-4t^2}\right)^2}{(2t)^2-\left(1-\sqrt{1-4t^2}\right)^2}\cdot\frac{(2t)^k+\left(1-\sqrt{1-4t^2}\right)^k}{(2t)^k-\left(1-\sqrt{1-4t^2}\right)^k},$$

or, expanding,

$$G(t)=\frac{(2t)^k+\sum_{m=0}^{k/2}\left(1-4t^2\right)^m\binom{k}{2m}}{\sum_{m=1}^{(k+1)/2}\left(1-4t^2\right)^m\binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u,t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.

For odd k, letting $k = 2\ell + 1$, we obtain

$$F(u,t) = \frac{1+(1-u)t}{1-(1+u)t} \cdot \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m (1+(1-u^2)t^2)^{\ell-m}}{\sum_{m=0}^{\ell} \alpha_m^\ell t^m (1+(1-u^2)t^2)^{\ell-m}},$$

$$G(t) = \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m}{(1-2t)\left(\sum_{m=0}^{\ell} \alpha_m^\ell t^m\right)},$$

where

$$\alpha_m^\ell = \begin{cases} (-)^{\frac{m}{2}} \binom{\ell - \frac{m}{2}}{\frac{m}{2}} & \text{if } m \equiv 0 \ [2], \\ (-)^{\frac{m-1}{2}} \binom{\ell - \frac{m+1}{2}}{\frac{m-1}{2}} & \text{if } m \equiv 1 \ [2]. \end{cases}$$

For even k, with $k = 2\ell$,

$$F(u,t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m (1-(1-u^2)t^2)^{\ell-2m}}{(1-(1+u)^2t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m (1-(1-u^2)t^2)^{\ell-1-2m}\right)},$$

$$G(t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m}{(1-4t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m\right)},$$

expressed as reduced fractions.

The first few values of F, where $\square$ stands for $1+(1-u^2)t^2$, are:

k	$F(u,t)$	k	$F(u,t)$
1	$\dfrac{1+(1-u)t}{1-(1+u)t}$	2	$\dfrac{\square}{1-(1+u)^2t^2}$
3	$\dfrac{(1+(1-u)t)(\square-t)}{(1-(1+u)t)(\square+t)}$	4	$\dfrac{\square^2-2t^2}{1-(1+u)^2t^2}$
5	$\dfrac{(1+(1-u)t)(\square^2-\square t-t^2)}{(1-(1+u)t)(\square^2+\square t+t^2)}$	6	$\dfrac{\square^2-3t^2}{(1-(1+u)^2t^2)(\square^2-t^2)}$
7	$\dfrac{(1+(1-u)t)(\square^3-\square^2t-2\square t^2+t^3)}{(1-(1+u)t)(\square^3+\square^2t-2\square t^2-t^3)}$	8	$\dfrac{\square^4-4\square^2t^2+2t^4}{(1-(1+u)^2t^2)(\square^2-2t^2)}$
9	$\dfrac{(1+(1-u)t)(\square-t)(\square^3-3\square t^2-t^3)}{(1-(1+u)t)(\square+t)(\square^3-3\square t^2+t^3)}$	10	$\dfrac{\square^4-5\square^2t^2+5t^4}{(1-(1+u)^2t^2)(\square^4-3\square^2t^2+t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*™.

7.3 Trees

Let $\mathcal{X}$ be the d-regular tree. Then

$$F(0,t)=1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u,t)$ and also obtain the answer using Corollary 2.6.

Let $\mathcal{T}$ be a regular tree of degree d with a fixed root $\star$, and let $\mathcal{T}'$ be the connected component of $\star$ in the two-tree forest obtained by deleting in $\mathcal{T}$ an edge at $\star$. Let $F(u,t)$ and $F'(u,t)$ respectively count circuits at $\star$ in $\mathcal{T}$ and $\mathcal{T}'$. For instance if $d=2$ then F' counts circuits in $\mathbf{N}$ and F counts circuits in $\mathbf{Z}$. For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F'. Then we have

$$F' = u + (d-1)\,tF't\frac{1}{1-(d-2+u)\,tF't},$$
$$F = 1 + dtF't\frac{1}{1-(d-1+u)\,tF't}.$$

Indeed a circuit in $\mathcal{T}'$ is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d-1$ directions, then a 'subcircuit' not returning to $\star$, then a step back to $\star$, followed by a step in any of $d-1$ directions (counting an extra factor of u if it was the same as before), a subcircuit, etc. If the 'subcircuit' is the empty circuit, it contributes a bump, hence the convention on F'. Likewise, a circuit in $\mathcal{T}$ is either the empty circuit (now counted as 1) or a sequence of circuits in subtrees each isomorphic to $\mathcal{T}'$.

We solve these equations to

$$F'(1-u,t) = \frac{2(1-u)}{1-u(d-u)t^2+\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}},$$

$$F(1-u,t) = \frac{2(d-1)(1-u^2t^2)}{(d-2)(1+u(d-u)t^2)+d\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}}.$$

Using (2.3) and $F(0,t)=1$ we would obtain

$$G(t) = \frac{1+(d-1)\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}{1-\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2},$$

or, after simplification,

$$G(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

which could have been obtained by setting $u=0$ in $F(1-u,t)$.

In particular if $d=2$, then $\mathcal{X}=C_\infty=\mathbf{Z}$ and

$$G(t) = \sum_{n\ge 0}\binom{2n}{n}t^{2n} = \frac{1}{\sqrt{1-4t^2}}.$$

Note that for all d the d-regular tree $\mathcal{X}$ is the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z})^{*d}$ with standard generating set. If d is even, $\mathcal{X}$ is also the Cayley graph of a free group of rank $d/2$ generated by a free set. We have thus computed the spectral radius of a random walk on a freely generated free group: it is, for $(\mathbf{Z}/2\mathbf{Z})^{*d}$ or for $\mathbf{F}_{d/2}$, equal to

$$\frac{2\sqrt{d-1}}{d}. \tag{7.1}$$

Remark that for $d=2$ the series $F(u,t)$ does have a simple series expansion. By direct expansion, we obtain the number of circuits of length $2n$ in $\mathbf{Z}$, with m local extrema, as

$$(t^{2n}u^m \mid F(u,t)) = \begin{cases} 2\binom{n-1}{\frac{m-1}{2}}^2 & \text{if } m\equiv 1\ [2],\\ 2\binom{n-1}{\frac{m}{2}}\binom{n-1}{\frac{m-2}{2}} & \text{if } m\equiv 0\ [2].\end{cases}$$

We may even look for a richer generating series than F: let

$$H(u,v,t)=\sum_{\pi:\ \text{path starting at } \star} u^{\mathrm{bc}(\pi)} v^{\delta(\star,\pi_{|\pi|})} t^{|\pi|} \in \mathbf{N}[u,v][[t]]\,,$$

where δ denotes the graph distance. Then

$$\begin{aligned} H(1,v,t) &= F(1,t) + dF'tvF + dF'tv(d-1)F'tvF + \dots \\ &= \frac{1+F'(1,t)tv}{1-(d-1)F'(1,t)tv} F(1,t)\,; \end{aligned}$$

and as H is a sum of series counting paths between fixed vertices we obtain $H(u,v,t)$ from $H(1,v,t)$ by extending (2.2) linearly:

$$\frac{H(1-u,v,t)}{1-u^2t^2} = \frac{H\left(1,v,\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}\,.$$

We could also have started by computing

$$H(0,v,t)=\frac{1+vt}{1-(d-1)vt}\,,$$

the growth series of all proper paths in $\mathcal{T}$, and using (2.3) and (2.5) obtain

$$H(1,v,t)=\frac{1+\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2t}\right)^2}{1-u^2\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}\cdot H\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t},0,v\right),$$

$$H(u,v,t)=\frac{1-t^2u^2}{1+u(d-u)t^2}\cdot\frac{(d-1)(4t^2+\Box^2)}{4(d-1)^2t^2-u^2\Box^2}\cdot\frac{2(d-1)t+v\Box}{2t-v\Box}\,,$$

where $\Box = 1+u(d-u)t^2-\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}$.

Recall that the growth series of a graph $\mathcal{X}$ at a base point $\star$ is the power series

$$P(t)=\sum_{v\in V(\mathcal{X})} t^{\delta(\star,v)}\,,$$

where δ denotes the distance in $\mathcal{X}$. The series H is very general in that it contains a lot of information on $\mathcal{T}$, namely

- $H(u,0,t)=F(u,t)$;
- $H(0,1,t)=\frac{1+t}{1-(d-1)t}=P(t)$ is the growth series of $\mathcal{T}$;
- $H(1,1,t)=1/(1-dt)$ is the growth series of all paths in $\mathcal{T}$.

(Note that these substitutions yield well-defined series because for any i there are only finitely many monomials having t-degree equal to i.)

We can also use this series H to compute the circuit series F_C of the cycle of length k, that was found in the previous section. Indeed the universal cover of a cycle is the regular tree $\mathcal{T}$ of degree 2, and circuits in C correspond bijectively to paths in $\mathcal{T}$ from $\star$ to any vertex at distance a multiple of k. We thus have

$$F_C(u,t) = \sum_{\zeta:\zeta^k=1} H(u,\zeta,t)$$

where the sum runs over all kth roots of unity and $d=2$ in H.

We consider next the following graphs: take a d-regular tree and fix a vertex $\star$. At $\star$, delete e vertices and replace them by e loops. Then clearly

$$F(0,t) = \frac{1+t}{1-(e-1)t},$$

as all the non-backtracking paths are constrained to the e loops. Using (2.3), we obtain after simplifications

$$G(t) = \frac{2(d-1)}{d+e-2-2e(d-1)t+(d-e)\sqrt{1-4(d-1)t^2}}. \tag{7.2}$$

The radius of convergence of G is

$$\min\left\{\frac{1}{2\sqrt{d-1}}, \frac{e-1}{d+e^2-2e}\right\}.$$

7.4 Tougher examples

In this subsection we outline the computations of F and G for more complicated graphs. They are only provided as examples and are logically independent from the remainder of the paper. The arguments will therefore be somewhat condensed.

First take for $\mathcal{X}$ the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}$ with generators $(0,-1) =$ '$\downarrow$', $(0,1) =$ '$\uparrow$' and $(1,0) =$ '$\leftrightarrow$'. Geometrically, $\mathcal{X}$ is a doubly-infinite two-poled ladder.

In Subsection 7.3 we computed

$$F_{\mathbf{Z}}(u,t) = \frac{1-(1-u)^2t^2}{\sqrt{(1+(1-u^2)t^2)^2-4t^2}},$$

the growth of circuits restricted to one pole of the ladder. A circuit in $\mathcal{X}$ is a circuit in $\mathbf{Z}$, before and after each step ($\uparrow$ or $\downarrow$) of which we may switch to the other pole (with a $\leftrightarrow$) as many times as we wish, subject to the condition that the circuit finish at the same pole as it started. This last condition is expressed by the fact that the series we obtain must have only coefficients of even degree in t. Thus, letting $\text{even}(f) = \frac{f(t)+f(-t)}{2}$, we have

$$G(t) = \mathrm{even}\Big(\frac{1}{1-t}F_{\mathbf{Z}}\Big(1, \frac{t}{1-t}\Big)\Big);$$

it is then simple to obtain $F(u,t)$ by performing the substitution (2.3).

The following direct argument also gives $F(u,t)$: a walk on the ladder is obtained from a walk on a pole (i.e. on $\mathbf{Z}$) by inserting before and after every step on a pole a (possibly empty) sequence of steps from one pole to the other. This process is expressed by performing on $F_{\mathbf{Z}}$ the substitution

$$t \mapsto t + t^2 + t^3u + t^4u^2 + \cdots = t + \frac{t^2}{1-tu},$$

corresponding to replacing a step on a pole by itself, or itself followed by a step to the other pole, or itself, a step to the other pole and a step back, etc. But if the path had a bump at the place the substitution was performed, this bump would disappear when a step is added from one pole to the other. In formulas,

$$tu \mapsto tu + t^2 + t^3u + t^4u^2 + \cdots = tu + \frac{t^2}{1-tu}.$$

Finally we must add at the beginning of the path a sequence of steps from one pole to the other. Therefore we obtain

$$F(u,t) = \mathrm{even}\Big\{\Big(1 + \frac{t}{1-tu}\Big)F_{\mathbf{Z}}\Big(\frac{tu + t^2/(1-tu)}{t + t^2/(1-tu)}, t + \frac{t^2}{1-tu}\Big)\Big\}.$$

As another example, consider the group $\mathbf{Z}$ generated by the non-free set $\{\pm 1, \pm 2\}$. Geometrically, it can be seen as the set of points $(2i, 0)$ and $(2i+1, \sqrt{3})$ for all $i \in \mathbf{Z}$, with edges between all points at Euclidean distance 2 apart; but we will not make use of this description. The circuit series of $\mathbf{Z}$ with this enlarged generating set will be an algebraic function of degree 4 over the rationals.

Define first the following series:

$f(t)$ counts the walks from 0 to 0 in $\mathbf{N}$;

$g(t)$ counts the walks from 0 to 1 in $\mathbf{N}$;

$h(t)$ counts the walks from 1 to 1 in $\mathbf{N}$.

Denote the generators of $\mathbf{Z}$ by $1 = \uparrow$, $2 = \upuparrows$, $-1 = \downarrow$ and $-2 = \downdownarrows$. The series then satisfy the following equations, where the generators' symbol is written instead of 't' to make the formulas self-explanatory:

$$f = 1 + (\uparrow f \downarrow + \uparrow g \downdownarrows + \upuparrows g \downarrow + \upuparrows h \downdownarrows) f,$$
$$g = f \uparrow f + f \upuparrows g,$$
$$h = f + f \downarrow g + g \downdownarrows g,$$

giving a solution f that is algebraic of degree 4 over $\mathbf{Z}(t)$.

Then define the following series:

G counts the walks from 0 to 0 in $\mathbf{Z}$;

e counts the walks from 0 to 1 in $\mathbf{Z}$.

They satisfy the equations

$$G = 1 + 2(\uparrow f \downarrow G + \upuparrows g \downarrow G + \uparrow f \downdownarrows e + \upuparrows g \downdownarrows e + \uparrow g \downdownarrows G + \upuparrows h \downdownarrows G),$$
$$e = G \uparrow f + G \upuparrows f + G \upuparrows g$$

giving the solution

$$G = \frac{4 + 3t - 6t^2 - 10t(1+2t)\delta + 2t^2(3+8t)\delta^2 - 6t^4(1+t)\delta^3}{4 - 7t - 36t^2}$$

where δ is a root of the equation

$$1 - (2t+1)\delta + t(2+3t)\delta^2 - t^2(1+2t)\delta^3 + t^4\delta^4 = 0\,.$$

8. Cogrowth of non-free presentations

We perform here a computation extending the results of Section 3.1. The general setting, expressed in the language of group theory, is the following: let Π be a group generated by a finite set S and let $\Xi < \Pi$ be any subgroup. We consider the following generating series:

$$F(t) = \sum_{\gamma \in \Xi < \Pi} t^{|\gamma|},$$

$$G(t) = \sum_{\substack{\text{words } w \text{ in } S \\ \text{defining an element in } \Xi}} t^{|w|},$$

where $|\gamma|$ is the minimal length of γ in the generators S, and $|w|$ is the usual length of the word w. Is there some relation between these series? In case Π is quasi-free on S, the relation between F and G is given by Corollary 2.6. We consider two other examples: Π quasi-free but on a set smaller than S, and $\Pi = \mathbf{PSL}_2(\mathbf{Z})$.

8.1 Π QUASI-FREE

Let S, T be finite sets, and $\bar{\cdot}$ an involution on S. Consider the two presentations

$$\Pi = \langle S \mid s\bar{s} = 1 \ \forall s \in S \rangle ,$$
$$\Pi = \langle S \cup T \mid s\bar{s} = 1 \ \forall s \in S; \ t = 1 \ \forall t \in T \rangle .$$

Let $\Xi < \Pi$ be any subgroup, and let F' and G' be the generating series related to the first presentation. Clearly $F' = F$, as both series count the same objects in Π (regardless of Π's presentation); while

$$G(t) = \frac{G'\left(\frac{t}{1-|T|t}\right)}{1 - |T|t} .$$

Indeed any word $w = w_1 \dots w_n$ in $S \cup T$ defining an element of Ξ can be uniquely decomposed as $w = t_0 s_1 t_1 \dots s_m t_m$, where $s_i \in S$, t_i are words in T for all i, and $s_1 \dots s_n$ defines an element of Ξ; moreover all choices of $s_1 \dots s_n$ defining an element of Ξ and words t_i in T give a distinct word w. It then suffices to note that the generating series for any of the t_i is $1/(1 - |T|t)$.

Putting everything together, we obtain:

PROPOSITION 8.1. *Let Π be as above, $\Xi < \Pi$ a subgroup. Then*

$$\frac{F(t)}{1 - t^2} = \frac{G\left(\frac{t}{1+|T|t+(|S|-1)t^2}\right)}{1 + |T|t + (|S| - 1)t^2} .$$

8.2 $\Pi = \mathbf{PSL}_2(\mathbf{Z})$

Let

$$\Pi = \mathbf{PSL}_2(\mathbf{Z}) = \langle a, b \mid a^2, b^3 \rangle ,$$

and let $\Xi < \Pi$ be any subgroup. We take $S = \{a, b, b^{-1}\}$.

We suppose Ξ is torsion-free, i.e. contains no element of the form waw^{-1} or $wb^{\pm 1}w^{-1}$. Let $\mathcal{X}$ be the Schreier graph of $(\Pi, \{a, b, b^{-1}\})$ relative to Ξ, as defined in Subsection 3.1; it is a trivalent graph whose vertex set is $\Xi \backslash \Pi$. Its vertices can be grouped in triples $w^\Delta = \{w, wb, wb^{-1}\}$ connected in triangles. Let $\mathcal{F}$ be the graph obtained from $\mathcal{X}$ by identifying each triple to a vertex. Explicitly,

$$V(\mathcal{F}) = \{w^\Delta : w \in V(\mathcal{X})\},$$
$$E(\mathcal{F}) = \{(v^\Delta, (va)^\Delta) : v \in V(\mathcal{X})\};$$

the involution on $E(\mathcal{F})$ is the switch $(A,B) \mapsto (B,A)$ and the extremity functions $E(\mathcal{F}) \to V(\mathcal{F})$ are the natural projections. Note that $\mathcal{F}$ is a 3-regular graph (for instance, 1^Δ is connected to a^Δ, $(ba)^\Delta$ and $(b^{-1}a)^\Delta$). In case $\Xi = 1$, it is the 3-regular tree. By construction we have a 3-to-1 map $\Delta\colon V(\mathcal{X}) \to V(\mathcal{F})$. We fix an origin $\star = 1^\Delta$ in $\mathcal{F}$, and let $F_{\mathcal{F}}(u,t)$ be the circuit series of $(\mathcal{F}, \star)$.

Let $\mathcal{E}$ be a triangle, $G_{\mathcal{E}}(t)$ count the circuits at a fixed vertex of $\mathcal{E}$ and $G^{\neq}_{\mathcal{E}}(t)$ count paths between two fixed distinct vertices of $\mathcal{E}$. These series were computed in Section 7.1, with $G^{\neq}_{\mathcal{E}}(t) = F'(1,t) + F''(1,t)$.

Circuits at $\star$ in $\mathcal{X}$ can be projected to circuits at $\star$ in $\mathcal{F}$ simply by deleting all edges of type $(w, wb^{\pm 1})$ and projecting the other edges through Δ. Conversely, circuits in $\mathcal{F}$ can be lifted to $\mathcal{X}$ by lifting the edges through Δ^{-1}, and connecting them in $\mathcal{X}$ with arbitrary paths remaining inside the triples: to lift the path $\pi = (\pi_1, \dots, \pi_n)$ from $\mathcal{F}$ to $\mathcal{X}$, choose edges $\rho_1, \dots, \rho_n$ with $(\rho_i^\alpha)^\Delta = \pi_i^\alpha$ and $(\rho_i^\omega)^\Delta = \pi_i^\omega$ for all $i \in \{1, \dots, n\}$, and choose, for all $i \in \{0, \dots, n\}$, paths τ_i from ρ_i^ω to ρ_{i+1}^α remaining inside $(\rho_i^\omega)^\Delta$, where by convention $\rho_0^\omega = \rho_{n+1}^\alpha = \star$. Then the lift corresponding to these choices is

$$\tau_0 \cdot \rho_1 \cdot \tau_1 \cdot \dots \cdot \rho_n \cdot \tau_n . \tag{8.1}$$

Furthermore all circuits at $\star$ in $\mathcal{X}$ can be obtained this way.

Define $\overline{G}$ as the series counting paths that start and finish at a vertex in the same triple as $\star$. It can be obtained using (8.1) by letting ρ range over all paths in $\mathcal{F}$, and for each choice of ρ and for each $i \in \{1, \dots, n-1\}$ letting τ_i range over $G_{\mathcal{E}}$ or $G^{\neq}_{\mathcal{E}}$ depending on whether ρ has or not a bump at i, and letting τ_0 and τ_n range over all paths inside the triple $\star^\Delta$. In equations, this relation is expressed as

$$\overline{G}(t) = \left(\frac{1}{1-2t}\right)^2 / G_{\mathcal{E}}(t) \cdot F_{\mathcal{F}}\left(G^{\neq}_{\mathcal{E}}(t)/G_{\mathcal{E}}(t), tG_{\mathcal{E}}(t)\right) .$$

Now the series G we wish to obtain is approximately $\overline{G}(t)/9$: for any choice of $x, y \in \star^\Delta$ there are approximately the same number of long enough paths from x to y.

A summand of $F(t)$ is the unique lifting of a summand of $F_{\mathcal{F}}(0,t)$, but is twice longer in $\mathcal{X}$ than in $\mathcal{F}$.

DEFINITION 8.2. Two series $A(t)$, $B(t)$ are *equivalent*, written $A \sim B$, if they have the same radius of convergence ρ, and there exists a constant K such that

$$\frac{1}{K} < A(t)/B(t) < K \text{ as } t \to \rho\,.$$

Then the remarks of the previous paragraph can be written as

$$F(t) \sim F_{\mathcal{F}}(0, t^2)\,,$$
$$G(t) \sim F_{\mathcal{F}}(G_{\mathcal{E}}^{\neq}(t)/G_{\mathcal{E}}(t), tG_{\mathcal{E}}(t))\,.$$

Letting $G_{\mathcal{F}}$ be the circuit series of $\mathcal{F}$, we use Corollary 2.6 to obtain

$$G_{\mathcal{E}}(t)^{\neq} = \frac{t}{1-t-2t^2}\,, \qquad G_{\mathcal{E}}(t) = \frac{1-t}{1-t-2t^2}\,,$$
$$F(t) \sim G_{\mathcal{F}}\Big(\frac{t^2}{1+2t^4}\Big)\,, \qquad G(t) \sim G_{\mathcal{F}}\Big(\frac{t^2}{1-t-3t^2}\Big)\,,$$

so

$$F(t) \sim G\Big(\frac{t\sqrt{4+13t^2-8t^4}-t^2}{2(1+t^2)(1+2t^2)}\Big)\,.$$

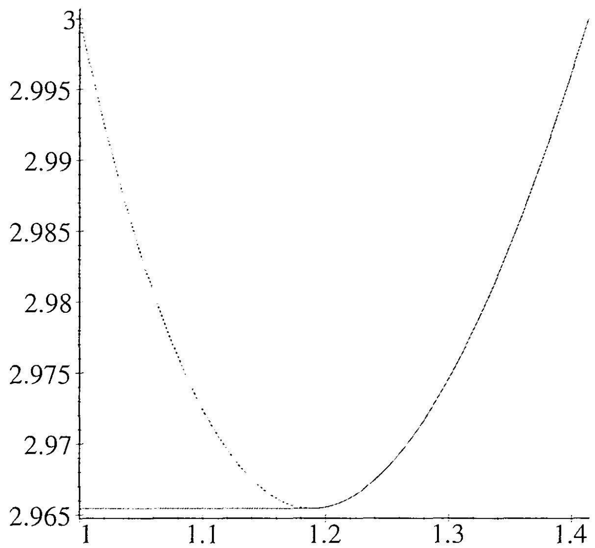

FIGURE 2

The function $\alpha \mapsto \nu$ relating cogrowth and spectral radius, for subgroups of $\mathbf{PSL}_2(\mathbf{Z})$

Let $\mathcal{X}$ be a simplicial complex such that at each vertex an edge and a (filled-in) triangle meet; choose a base point $\star$ in $\mathcal{X}$. Say a circuit in the 1-skeleton of $\mathcal{X}$ is *reduced* if it contains no bump nor two successive edges in the same triangle; thus reduced circuits are in bijection with homotopy classes in $\pi_1(\mathcal{X},\star)$. Let $F(t)$ be the proper circuit series and $G(t)$ the circuit series of $\mathcal{X}$. Let

$$(t)\phi = \frac{t\sqrt{4+13t^2+8t^4}-t^2}{2(1+t^2)(1+2t^2)} .$$

We have proved the following theorem and corollary, similar to those in Section 3.1:

THEOREM 8.3. $F(t) \sim G\big((t)\phi\big)$.

COROLLARY 8.4. *Let Ξ be a subgroup of $\Pi = \mathbf{PSL}_2(\mathbf{Z})$; let ν be the spectral radius of the simple random walk on $\Xi\backslash\Pi$, and α the "cogrowth" rate of $\Xi\backslash\Pi$. Then provided that $\alpha \in [\sqrt{\rho},\rho]$, where ρ is the word growth of Π, namely $\sqrt{2}$, we have*

$$1/\nu = (1/\alpha)\phi, \quad \textit{so} \quad \nu = \frac{1}{2}\sqrt{8\alpha^{-2}+13+4\alpha^2}+\frac{1}{2} .$$

Proof. The function ϕ is monotonously increasing between 0 and $1/\sqrt[4]{2}$, where it reaches its maximum. The same argument applies as that given in the proof of Corollary 3.2. □

We now state the same results for an arbitrary virtually free group with an appropriate generating system. Let Π be a virtually free group, such that there is a split exact sequence

$$1 \longrightarrow \Sigma \longrightarrow \Pi \overset{\pi}{\rightleftarrows} \Upsilon \longrightarrow 1$$

where Υ is a finite group and Σ has a presentation

$$\Sigma = \langle s \in S \mid s\bar{s} = 1 \quad \forall s \in S\rangle .$$

We assume further that Π is generated by a set $T = T' \sqcup T''$ with T'' in bijection through π with $\Upsilon \setminus \{1\}$, T' mapping through π to $\{1\}$, and $T' \times (T'' \cup \{1\})$ in bijection with S through $(t,p) \mapsto p^{-1}tp$.

For example, consider $\Pi = \mathbf{PSL}_2(\mathbf{Z}) = \langle a,b,b^{-1}\rangle$. Take $T' = \{a\}$ and $T'' = \{b,b^{-1}\}$, take $\Upsilon = \langle b,b^{-1}\rangle$ and $\Sigma = \langle a, bab^{-1}, b^{-1}ab\rangle$. Then the hypotheses are satisfied.

With these hypotheses, the Cayley graph $\mathcal{X}$ of Π is a collection of complete graphs of size $|\Upsilon|$, with at each vertex $|T'|$ edges leaving to other complete graphs, and such that if each of these complete graphs is shrunk to a point the resulting graph is a tree. The following theorem is then a straightforward generalization of the argument given for $\mathbf{PSL}_2(\mathbf{Z})$.

THEOREM 8.5. *With the notation introduced above, let Ξ be any subgroup of* Π *not intersecting* $\{t^\gamma \mid t \in T, \gamma \in \Pi\}$ *and let* $F(t)$, $G(t)$ *be the "cogrowth" series and circuit series of* $\Xi\backslash\Pi$. *Let* $\mathcal{E}$ *be the complete graph on* $|\Upsilon|$ *vertices and let* $G_{\mathcal{E}}(t)$, $G^{\neq}_{\mathcal{E}}(t)$ *count the circuits and the non-closing paths respectively in* $\mathcal{E}$. *Define the function* ϕ *by*

$$\Big(\frac{t^2}{1+(|S|-1)t^4}\Big)\phi = \frac{tG_{\mathcal{E}}}{1+(G_{\mathcal{E}}-G^{\neq}_{\mathcal{E}})((|S|-1)G_{\mathcal{E}}+G^{\neq}_{\mathcal{E}})t^2}\,.$$

Then we have

$$F(t) \sim G((t)\phi)\,.$$

9. FREE PRODUCTS OF GRAPHS

We give here a general construction combining two pointed graphs and show how to compute the generating functions for circuits in the "product" in terms of the generating functions for circuits in the factors.

DEFINITION 9.1 (Free Product, [Que94, Definition 4.8]). Let $(\mathcal{E},\star)$ and $(\mathcal{F},\star)$ be two connected pointed graphs. Their *free product* $\mathcal{E}*\mathcal{F}$ is the graph constructed as follows: start with copies of $\mathcal{E}$ and $\mathcal{F}$ identified at $\star$; at each vertex v apart from $\star$ in $\mathcal{E}$, respectively $\mathcal{F}$, glue a copy of $\mathcal{F}$, respectively $\mathcal{E}$, by identifying v and the $\star$ of the copy. Repeat the process, each time glueing $\mathcal{E}$'s and $\mathcal{F}$'s to the new vertices.

If (E,S), (F,T) are two groups with fixed generators whose Cayley graphs are $\mathcal{E}$ and $\mathcal{F}$ respectively, then $\mathcal{E}*\mathcal{F}$ is the Cayley graph of $(E*F, S\sqcup T)$.

We now compute the circuit series of $\mathcal{E}*\mathcal{F}$ in terms of the circuit series of $\mathcal{E}$ and $\mathcal{F}$. Let $G_{\mathcal{E}}$, $G_{\mathcal{F}}$ and $G_{\mathcal{X}}$ be the generating functions counting circuits in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{X}=\mathcal{E}*\mathcal{F}$ respectively. We will use the following description: given a circuit at $\star$ in $\mathcal{X}$, it can be decomposed as a product of circuits never passing through $\star$. Each of these circuits, in turn, starts either in the $\mathcal{E}$ or the $\mathcal{F}$ copy at $\star$. Say one starts in $\mathcal{E}$; it can then be expressed as a circuit in $\mathcal{E}$ never passing through $\star$, and such that at all vertices, except the first and last, a circuit starting in $\mathcal{F}$ has been inserted. Moreover, any choice of such circuits satisfying these conditions will give a circuit at $\star$ in $\mathcal{X}$, and different choices will yield different circuits.

Let $H_{\mathcal{E}}$ (respectively $H_{\mathcal{F}}$) be the generating function counting non-trivial circuits in $\mathcal{E}$ (respectively $\mathcal{F}$) never passing through $\star$. Obviously

$$G_{\mathcal{E}} = \frac{1}{1 - H_{\mathcal{E}}} \quad \text{so} \quad H_{\mathcal{E}} = 1 - \frac{1}{G_{\mathcal{E}}}.$$

Let $L_{\mathcal{E}}$ (respectively $L_{\mathcal{F}}$) be the generating function counting non-trivial circuits in $\mathcal{X}$ never passing through $\star$ and starting in $\mathcal{E}$ (respectively $\mathcal{F}$). Then

$$L_{\mathcal{E}}(t) = H_{\mathcal{E}}\Big(\frac{t}{1 - L_{\mathcal{F}}(t)}\Big) \cdot (1 - L_{\mathcal{F}}(t)),$$
$$L_{\mathcal{F}}(t) = H_{\mathcal{F}}\Big(\frac{t}{1 - L_{\mathcal{E}}(t)}\Big) \cdot (1 - L_{\mathcal{E}}(t)).$$

Indeed write $H_{\mathcal{E}} = \sum h_n t^n$. Then by the description given above

$$L_{\mathcal{E}} = \sum h_n t^n \Big(\frac{1}{1 - L_{\mathcal{F}}}\Big)^{n-1},$$

which is precisely the given formula. Finally

$$G_{\mathcal{X}} = \frac{1}{1 - L_{\mathcal{E}} - L_{\mathcal{F}}}.$$

Writing $M_{\mathcal{E}} = t/(1 - L_{\mathcal{E}})$ and $M_{\mathcal{F}} = t/(1 - L_{\mathcal{F}})$, we simplify these equations to

$$1 - \frac{t}{M_{\mathcal{E}}} = \Big(1 - \frac{1}{G_{\mathcal{E}}(M_{\mathcal{F}})}\Big) \cdot \frac{t}{M_{\mathcal{F}}},$$
$$1 - \frac{t}{M_{\mathcal{F}}} = \Big(1 - \frac{1}{G_{\mathcal{F}}(M_{\mathcal{E}})}\Big) \cdot \frac{t}{M_{\mathcal{E}}},$$
$$G_{\mathcal{X}} = \frac{1}{1 - \big(1 - \frac{t}{M_{\mathcal{E}}}\big) - \big(1 - \frac{t}{M_{\mathcal{F}}}\big)} = \frac{1/t}{1/M_{\mathcal{E}} + 1/M_{\mathcal{F}} - 1/t},$$

so

$$\frac{1}{M_{\mathcal{E}}} + \frac{1}{M_{\mathcal{F}}} - \frac{1}{t} = \frac{1}{M_{\mathcal{E}} G_{\mathcal{F}}(M_{\mathcal{E}})} = \frac{1}{M_{\mathcal{F}} G_{\mathcal{E}}(M_{\mathcal{F}})} = \frac{1}{tG_{\mathcal{X}}}. \tag{9.1}$$

If f is a power series with $f(0) = 0$ and $f'(0) \neq 0$, let us write f^{-1} for the inverse of f; i.e. for the series g with $g(f(t)) = f(g(t)) = t$ (for instance, $f(t) = t$ is equal to its inverse; the inverse of $\frac{at+b}{ct+d}$ is $\frac{dt-b}{-ct+a}$).

From (9.1) we obtain $M_{\mathcal{E}} = (tG_{\mathcal{F}})^{-1} \circ (tG_{\mathcal{X}})$ and $M_{\mathcal{F}} = (tG_{\mathcal{E}})^{-1} \circ (tG_{\mathcal{X}})$; so composing (9.1) with $(tG_{\mathcal{X}})^{-1}$ we obtain the

THEOREM 9.2.

$$\frac{1}{(tG_{\mathcal{X}})^{-1}} = \frac{1}{(tG_{\mathcal{E}})^{-1}} + \frac{1}{(tG_{\mathcal{F}})^{-1}} - \frac{1}{t}\,. \tag{9.2}$$

An equation equivalent to this one, though not trivially so, appeared in a paper by Gregory Quenell [Que94], and, in yet another language, in a paper by Dan Voiculescu [Voi90, Theorem 4.5].

We can use (9.2) to obtain by a different method the circuit series of regular trees (see Section 7.3). Indeed the free product of regular trees of degree d and e is a regular tree of degree $d+e$. Letting G_d denote the circuit series of a regular tree of degree d, we "guess" that

$$G_d(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}}\,,$$

and verify that the limit

$$G_1(t) = \lim_{d\to 1} \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}} = \frac{1}{1-t^2}$$

is indeed the circuit series of the 1-regular tree. Then we compute

$$(tG_d)^{-1}(u) = \frac{2u}{2-d+d\sqrt{1+4u^2}}\,;$$

if we define $\triangle_d$ by

$$\triangle_d := \frac{1}{(tG_d)^{-1}(u)} - \frac{1}{u} = d\frac{\sqrt{1+4u^2}-1}{2u}\,,$$

it satisfies $\triangle_d + \triangle_e = \triangle_{d+e}$ and we have proved that our guess of G_d is correct for all $d \geq 1$, in light of (9.2).

As another application of (9.2), we compute the circuit series $G(t)$ of the Cayley graph of $\mathbf{PSL}_2(\mathbf{Z}) = \langle a, b \mid a^2, b^3 \rangle$ with generators $\{a, b, b^{-1}\}$. This graph is the free product of the 1-regular tree $\mathcal{E}$ and of the 3-cycle $\mathcal{F}$. We know from Section 7.1 that

$$G_{\mathcal{E}} = \frac{1}{1-t^2}\,, \qquad G_{\mathcal{F}} = \frac{1-t}{(1+t)(1-2t)}$$

are the circuit series of $\mathcal{E}$ and $\mathcal{F}$. We then compute

$$(tG_{\mathcal{E}})^{-1}(u) = \frac{\sqrt{1+4u^2}-1}{2u}\,,$$

$$(tG_{\mathcal{F}})^{-1}(u) = \frac{1+u-\sqrt{1-2u+9u^2}}{2(1-2u)}\,,$$

so after some lucky simplifications

$$G(t) = \frac{1}{t}\left(\frac{1}{1/(tF_{\mathcal{E}})^{-1}(u) + 1/(tG_{\mathcal{F}})^{-1}(u) - 1/u}\right)^{-1}$$
$$= \frac{(2-t)\sqrt{1-2t-5t^2+6t^3+t^4} - t + t^2 + t^3}{2(1-2t-5t^2+6t^3)} .$$

(A closed form such as this one does not exist for $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/k\mathbf{Z})$ with $k > 3$, because then the series G is algebraic of degree greater than 2.)

COROLLARY 9.3. *If the circuit series of $\mathcal{E}$ and $\mathcal{F}$ are both algebraic, then the circuit series of $\mathcal{E} * \mathcal{F}$ is also algebraic.*

Proof. Sums and products of algebraic series are algebraic. If f satisfies the algebraic relation $P(f, t) = 0$, then its formal inverse satisfies the relation $P(t, f^{-1}) = 0$ so is also algebraic. $\square$

Recall the notions of radius of convergence and ρ-recurrence given in Definition 3.4.

LEMMA 9.4. *We have*

$$\rho(f) = \sup_t (tf(t))^{-1} ,$$

where the supremum is taken over all t such that the series $(tf)^{-1}$ converges. If f is ρ-recurrent, then also

$$\rho(f) = \lim_{t \to \infty} (tf(t))^{-1} .$$

Proof. Clearly $\rho(f) = \rho(tf)$; if tf converges over $[0, \rho[$ then $(tf)^{-1}$ converges over $[0, \sigma[$ where $\sigma = \rho f(\rho)$; then we have $\lim_{t\to\sigma}(tf)^{-1} = \rho$. The second assertion follows because in this case $\sigma = \infty$. $\square$

COROLLARY 9.5. *Let the circuit series of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{X} = \mathcal{E} * \mathcal{F}$ be $G_{\mathcal{E}}$, $G_{\mathcal{F}}$ and $G_{\mathcal{X}}$ respectively, and suppose all three series are recurrent. Then*

$$1/\rho(G_{\mathcal{X}}) = 1/\rho(G_{\mathcal{E}}) + 1/\rho(G_{\mathcal{F}}) .$$

Proof. This follows from

$$\begin{aligned}1/\rho(G_{\mathcal{X}}) &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{X}})^{-1}}\\ &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{E}})^{-1}} + \frac{1}{(tG_{\mathcal{F}})^{-1}} - \frac{1}{t} \qquad \text{by (9.2)}\\ &= 1/\rho(G_{\mathcal{E}}) + 1/\rho(G_{\mathcal{F}}) - 0\,. \qquad \square\end{aligned}$$

Note that the corollary does not extend to non-recurrent series; for instance, it fails if $\mathcal{E} = \mathcal{F} = \mathbf{Z}$. Indeed then

$$G_{\mathcal{E}} = G_{\mathcal{F}} = \frac{1}{\sqrt{1-4t^2}}\,, \qquad \rho(G_{\mathcal{E}}) = \rho(G_{\mathcal{F}}) = 1/4\,,$$

$$G_{\mathcal{X}} = \frac{3}{1+2\sqrt{1-12t^2}}\,, \qquad \rho(G_{\mathcal{X}}) = 1/\sqrt{12}\,.$$

10. Direct products of graphs

There are two natural definitions for *direct products* of graphs; they correspond to direct products of groups with generating set either the union or cartesian product of the generating sets of the factors. A general treatment of products of graphs can be found in [CDS79, pages 65 and 203].

DEFINITION 10.1. If S is a set, the *stationing graph* on S is the graph $\mathcal{X} = \Sigma_S$ with $V(\mathcal{X}) = E(\mathcal{X}) = S$, where for the edges $s^\alpha = s^\omega = \bar{s} = s$ hold.

LEMMA 10.2. *Let $\mathcal{X}$ be a graph, and $\mathcal{E} = \mathcal{X} \sqcup \Sigma_{\mathcal{X}}$ be the graph obtained by adding a loop to every vertex in $\mathcal{X}$. Let $G_{\mathcal{X}}$ and $G_{\mathcal{E}}$ be the growth functions for circuits in $\mathcal{X}$ and $\mathcal{E}$ respectively. Then we have*

$$G_{\mathcal{E}}(t) = \frac{1}{1-t} G_{\mathcal{X}}\left(\frac{t}{1-t}\right).$$

DEFINITION 10.3 (First Product). Let $\mathcal{E}$ and $\mathcal{F}$ be two graphs. Their *direct product* $\mathcal{X} = \mathcal{E} \times \mathcal{F}$ is defined by

$$V(\mathcal{X}) = V(\mathcal{E}) \times V(\mathcal{F})$$

and

$$E(\mathcal{X}) = (E(\mathcal{E}) \times \Sigma_{\mathcal{F}}) \sqcup (\Sigma_{\mathcal{E}} \times E(\mathcal{F}))\,.$$

Note that if the graphs $\mathcal{E}$ and $\mathcal{F}$ have respectively adjacency matrices E and F, then their product has adjacency matrix $E \otimes \mathbf{1} + \mathbf{1} \otimes F$.

In that case we have

$$G_{\mathcal{X}} = \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{G_{\mathcal{E}}\left((1+u)t\right) G_{\mathcal{F}}\left((1+u^{-1})t\right)}{u} du .$$

This is a simple application of the Laplace transform, that converts an exponential generating function into an ordinary one and vice versa [AS70, 29.3.3]. Indeed, if we had considered exponential generating functions, the formula would simply have been $G_{\mathcal{X}} = G_{\mathcal{E}} G_{\mathcal{F}}$, as is well known (see [Wil90] or [Sta78, page 102]).

As an example, let $\mathcal{E} = \mathcal{F} = \mathbf{Z}$, so $G_{\mathcal{E}} = G_{\mathcal{F}} = \frac{1}{\sqrt{1-t^2}}$. Then

$$\begin{aligned} G_{\mathcal{X}} &= \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{du}{\sqrt{(1-4(1+u)^2t^2)(u^2-4(1+u)^2t^2)}} \\ &= \frac{2}{\pi} K(16t^2) = F\left(\begin{matrix} 1/2 & & 1/2 \\ & 1 & \end{matrix} \,\middle|\, 16t^2 \right) \end{aligned}$$

where K is the complete elliptic function and F the hypergeometric series. These functions are known to be transcendental; thus the circuit series of $\mathbf{Z}^2$ is transcendental. This result appears in [GH97]. Numerical evidence suggests the growth function for $\mathbf{Z}^3$ is not even hypergeometric.

DEFINITION 10.4 (Second Product). Let $\mathcal{E}$ and $\mathcal{F}$ be two graphs, and suppose that for every vertex in $\mathcal{E}$ and $\mathcal{F}$ there is a loop at it. Then their *direct product* $\mathcal{X} = \mathcal{E} \times \mathcal{F}$ is defined by

$$V(\mathcal{X}) = V(\mathcal{E}) \times V(\mathcal{F})$$

and

$$E(\mathcal{X}) = E(\mathcal{E}) \times E(\mathcal{F}) .$$

Note that if the graphs $\mathcal{E}$ and $\mathcal{F}$ have respectively adjacency matrices E and F, then their product has adjacency matrix $E \otimes F$.

In that case we have, again using Laplace transformations

$$G_{\mathcal{X}}(t) = \frac{1}{2i\pi} \oint_{\mathbf{S}^1} \frac{G_{\mathcal{E}}(u) G_{\mathcal{F}}(t/u)}{u} du .$$

Note that with both definitions of products it is possible that the growth function for circuits in the product be transcendental even if the growth functions for circuits in the factors are algebraic.

11. Further work

It was mentioned in Subsection 3.3 how the main result applies to languages. This a special case of a much more general problem:

PROBLEM 11.1. *Given a language* L *and a set* $\mathcal{U}$ *of words, define the desiccation* $L_{\mathcal{U}}$ *of* L *as the set of words in* L *containing no* $u \in \mathcal{U}$ *as a subword.*

Give sufficient conditions on L *and* $\mathcal{U}$ *such that a formula exist relating* $\Theta(L)$ *and* $\Theta(L_{\mathcal{U}})$.

The special case we studied in this paper is that of

$$\mathcal{U} = \{s\bar{s} \mid s \in S\}$$

and a sufficient condition is that L be saturated.

For general $\mathcal{U}$ this is not always sufficient: let $S = \{a, b\}$ and $L = b^*(ab^*ab^*)^*$ be the set of words with an even number of a's. Then if $\mathcal{U} = \{a^2\}$ there are 7 desiccated words of length 5:

$$\{b^5, ab^3a, ab^2ab, abab^2, bab^2a, babab, b^2aba\}$$

and if $\mathcal{U}' = \{b^2\}$ there are 6 desiccated words of length 5:

$$\{babab, ba^4, aba^3, a^2ba^2, a^3ba, a^4b\}\,.$$

The growth series of $\mathcal{U}$ and $\mathcal{U}'$ are the same, namely t^2, but the growth series of $L_{\mathcal{U}}$ and $L_{\mathcal{U}'}$ differ in their degree-5 coefficient.

We gave in Section 9 a formula relating the circuit series of a free product to the circuit series of its factors. There is a notion of *amalgamated product* of graphs, that is a direct analogue of the amalgamated product of groups.

PROBLEM 11.2. *What conditions on* $\mathcal{D}, \mathcal{E}, \mathcal{F}$ *are sufficient so that*

$$\frac{1}{(zG_{\mathcal{X}})^{-1}} = \frac{1}{(zG_{\mathcal{E}})^{-1}} + \frac{1}{(zG_{\mathcal{F}})^{-1}} - \frac{1}{(zG_{\mathcal{D}})^{-1}}$$

where $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ *is an amalgamated product of* $\mathcal{E}$ *and* $\mathcal{F}$ *along* $\mathcal{D}$ *?*

The formula holds if $\mathcal{D}$ is the trivial graph; but it cannot hold in general. If $\mathcal{E} = \mathcal{F}$ is the "ladder graph" described in Section 7.4: the set of points (i,j) with $i \in \mathbf{Z}$ and $j \in \{0,1\}$, with edges connecting all pairs of vertices at Euclidean distance 1, and $\mathcal{D}$ is $\mathbf{Z}$, embedded as a pole of the ladder, then the amalgamated product $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is isomorphic to $\mathbf{Z}^2$. The circuit series of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ have been calculated explicitly and are algebraic. The circuit series of $\mathcal{X}$ was shown in Section 10 to be transcendental; so there can exist no algebraic definition of $G_{\mathcal{X}}$ in terms of $G_{\mathcal{D}}$, $G_{\mathcal{E}}$ and $G_{\mathcal{F}}$. However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph $\mathcal{X}$, one can construct a graph $\mathcal{X}^{(k)}$ on the same vertex set, and with edge set the set of paths of length $\leq k$ in $\mathcal{X}$. Is there some simple relation between the path series of $\mathcal{X}$ and of $\mathcal{X}^{(k)}$? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's R-transform [Voi90]. His S-transform, in terms of graphs, corresponds to $\mathcal{E} * \mathcal{F}$ with as edge set all sequences (e,f) and (f,e), for $e \in E(\mathcal{E})$ and $f \in E(\mathcal{F})$. Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

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Added in proof. Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at http://www.math.berkeley.edu/~vfr/plnalg1.ps).

REFERENCES

[AS70] ABRAMOWITZ, M. and I. A. STEGUN. *Handbook of Mathematical Functions* (ninth edition). Dover, New York, 1970.

[Ami80] AMITSUR, S. A. On the characteristic polynomial of a sum of matrices. *Linear and Multilinear Algebra 8* (1980), 177–182.

[Bas92] BASS, H. The Ihara-Selberg zeta function of a tree lattice. *Internat. J. Math. 3* (1992), 717–797.

[Car92] CARTWRIGHT, D. I. Singularities of the Green function of a random walk on a discrete group. *Monatsh. Math. 113* (1992), no. 3, 183–188.

[Cha93] CHAMPETIER, C. Cocroissance des groupes à petite simplification. *Bull. London Math. Soc. 25* (1993), 438–444.

[Coh82] COHEN, J. M. Cogrowth and amenability of discrete groups. *J. Funct. Anal. 48* (1982), no. 3, 301–309.

[CP96] COORNAERT, M. and A. PAPADOPOULOS. Récurrence de marches aléatoires et ergodicité du flot géodésique sur les graphes réguliers. *Math. Scand. 79* (1996), no. 1, 130–152.

[CDS79] CVETKOVIĆ, D. M., M. DOOB, and H. SACHS. *Spectra of Graphs*. Pure and Applied Mathematics, Academic Press, 1979.

[Eil74] EILENBERG, S. *Automata, Languages, and Machines*, vol. A. Academic Press, 1974.

[FZ98] FOATA, D. and D. ZEILBERGER. A combinatorial proof of Bass's evaluations of the Ihara-Selberg zeta function for graphs. To appear in *Trans. Amer. Math. Soc.* 1998.

[God93] GODSIL, C. D. *Algebraic Combinatorics*. Chapman and Hall, New York, 1993.

[Gou72] GOULD, H. W. *Combinatorial Identities*. Morgantown Publishers, 1972.

[GKP94] GRAHAM, R. L., D. E. KNUTH, and O. PATASHNIK. *Concrete Mathematics* (2nd edition). Addison-Wesley, Reading, Mass., 1994.

[Gri78a] GRIGORCHUK, R. I. Banach invariant means on homogeneous spaces and random walks. PhD thesis (in Russian). Moscow State University, 1978.

[Gri78b] —— *Symmetric random walks on discrete groups. Multi-Component Random Systems* (R. L. Dobrushin and Y. G. Sinai, eds.), Nauka, Moscow, 1978. English translation: *Adv. Probab. Rel. Top. (D. Griffeath ed.) 6* M. Dekker 1980, 285–325, pp. 132–152.

[GH97] GRIGORCHUK, R. I. and P. DE LA HARPE. On problems related to growth, entropy, and spectrum in group theory. *J. of Dynamical and Control Systems 3* (1997), no. 1, 51–89.

[Kes59] KESTEN, H. Symmetric random walks on groups. *Trans. Amer. Math. Soc. 92* (1959), 336–354.

[Kit98] KITCHENS, B. P. *Symbolic Dynamics*. Springer-Verlag, 1998.

[Lia96] LIARDET, F. Croissance des groupes virtuellement abéliens. PhD thesis. University of Geneva, 1996.

[MW89] MOHAR, B. and W. WOESS. A survey on spectra of infinite graphs. *Bull. London Math. Soc. 21* (1989), no. 3, 209–234.

[MS81] MÜLLER, D. E. and P. E. SCHUPP. Context-free languages, groups, the theory of ends, second-order logic, tiling problems, cellular automata, and vector addition systems. *Bull. Amer. Math. Soc. (N.S.) 4* (1981), 331–334.

[MS83] MÜLLER, D. E. and P. E. SCHUPP. Groups, the theory of ends, and context-free languages. *J. Comput. System Sci. 26* (1983), 295–310.

[Nor92] NORTHSHIELD, S. Cogrowth of regular graphs. *Proc. Amer. Math. Soc. 116* (1992), no. 1, 203–205.

[Nor96] —— Proof of Ihara's theorem for regular and irregular graphs. *I.M.A. Workshop "Emerging Applications of Number Theory"*, 1996.

[Pas93] PASCHKE, W. L. Lower bound for the norm of a vertex-transitive graph. *Math. Z. 213* (1993), no. 2, 225–239.

[Que94] QUENELL, G. *Combinatorics of Free Product Graphs. Geometry of the Spectrum.* (Seattle, WA, 1993), Amer. Math. Soc., Providence, RI, 1994, pp. 257–281.

[RS87] REUTENAUER, C. and M.-P. SCHÜTZENBERGER. A formula for the determinant of a sum of matrices. *Lett. Math. Phys. 13* (1987), 299–302.

[Sta78] STANLEY, R. P. *Generating Functions. Studies in Combinatorics.* (Gian-Carlo Rota, ed.), MAA Studies in Mathematics, Volume 17, 1978, pp. 100–141.

[Szw89] SZWARC, R. A short proof of the Grigorchuk-Cohen cogrowth theorem. *Proc. Amer. Math. Soc. 106* (1989), no. 3, 663–665.

[Voi90] VOICULESCU, D. Noncommutative random variables and spectral problems in free product C^*-algebras. *Rocky Mountain J. Math. 20* (1990), no. 2, 263–283.

[Wil90] WILF, H. S. *Generatingfunctionology.* Academic Press Inc., Boston, MA, 1990.

[Woe83] WOESS, W. Cogrowth of groups and simple random walks. *Arch. Math. (Basel) 41* (1983), 363–370.

[Woe94] —— Random walks on infinite graphs and groups — a survey on selected topics. *Bull. London Math. Soc. 26* (1994), 1–60.

[Woe98] —— *Random Walks on Infinite Graphs and Groups.* Cambridge Tracts in Mathematics, in preparation (1998).

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