Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Autor: Bartholdi, Laurent
Kapitel: 1. Introduction
DOI: https://doi.org/10.5169/seals-64442

L'Enseignement Mathématique, t. 45 (1999), p. 83–131

COUNTING PATHS IN GRAPHS

by Laurent BARTHOLDI

ABSTRACT. We give a simple combinatorial proof of a formula that extends a result by Grigorchuk [Gri78a, Gri78b] relating cogrowth and spectral radius of random walks. Our main result is an explicit equation determining the number of 'bumps' on paths in a graph: in a d-regular (not necessarily transitive) non-oriented graph let the series $G(t)$ count all paths between two fixed points weighted by their length t^{length}, and $F(u,t)$ count the same paths, weighted as $u^{\text{number of bumps}} t^{\text{length}}$. Then one has

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2} .$$

We then derive the circuit series of 'free products' and 'direct products' of graphs. We also obtain a generalized form of the Ihara-Selberg zeta function [Bas92, FZ98].

1. INTRODUCTION

Let $\Gamma = \mathbf{F}_S/N$ be a group generated by a finite set S, where $\mathbf{F}_S$ denotes the free group on S. Let f_n be the number of elements of the normal subgroup N of $\mathbf{F}_S$ whose minimal representation as words in $S \cup S^{-1}$ has length n; let g_n be the number of (not necessarily reduced) words of length n in $S \cup S^{-1}$ that evaluate to 1 in Γ; and let $d = |S \cup S^{-1}| = 2|S|$. The numbers

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{f_n} , \qquad \nu = \frac{1}{d} \limsup_{n\to\infty} \sqrt[n]{g_n}$$

are called the *cogrowth* and *spectral radius* of (Γ, S). The Grigorchuk Formula [Gri78b] states that

$$(1.1) \qquad \nu = \begin{cases} \frac{\sqrt{d-1}}{d}\left(\frac{\alpha}{\sqrt{d-1}} + \frac{\sqrt{d-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{d-1} , \\ \frac{2\sqrt{d-1}}{d} & \text{else} . \end{cases}$$

We generalize this result to a somewhat more general setting: we replace the group Γ by a regular graph $\mathcal{X}$, i.e. a graph with the same number of edges at each vertex. Fix a vertex $\star$ of $\mathcal{X}$; let g_n be the number of circuits (closed sequences of edges) of length n at $\star$ and let f_n be the number of circuits of length n at $\star$ with no backtracking (no edge followed twice consecutively). Then the same equation holds between the growth rates of f_n and g_n.

To a group Γ with fixed generating set one associates its Cayley graph $\mathcal{X}$ (see Subsection 3.1). $\mathcal{X}$ is a d-regular graph with distinguished vertex $\star = 1$; paths starting at $\star$ in $\mathcal{X}$ are in one-to-one correspondence with words in $S \cup S^{-1}$, and paths starting at $\star$ with no backtracking are in one-to-one correspondence with elements of $\mathbf{F}_S$. A circuit at $\star$ in $\mathcal{X}$ is then precisely a word evaluating to 1 in Γ, and a circuit without backtracking represents precisely one element of N. In this sense results on graphs generalize results on groups. The converse would not be true: there are even graphs with a vertex-transitive automorphism group that are not the Cayley graph of a group [Pas93].

Even more generally, we will show that, rather than counting circuits and proper circuits (those without backtracking) at a fixed vertex, we can count paths and proper paths between two fixed vertices and obtain the same formula relating their growth rates.

These relations between growth rates are consequences of a stronger result, expressed in terms of generating functions. Define the formal power series

$$F(t) = \sum_{n=0}^{\infty} f_n t^n , \qquad G(t) = \sum_{n=0}^{\infty} g_n t^n .$$

Then assuming $\mathcal{X}$ is d-regular we have

$$\frac{F(t)}{1-t^2} = \frac{G\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} . \tag{1.2}$$

This equation relates F and G, and so relates *a fortiori* their radii of convergence, which are $1/\alpha$ and $1/(d\nu)$. We re-obtain thus the Grigorchuk Formula.

Finally, rather than counting paths and proper paths between two fixed vertices, we can count, for each $m \geq 0$, the number of paths with m backtrackings, i.e. with m occurrences of an edge followed twice in a row. Letting $f_{m,n}$ be the number of paths of length n with m backtrackings, consider the two-variable formal power series

$$F(u,t) = \sum_{m,n=0}^{\infty} f_{m,n} u^m t^n .$$

Note that $F(0,t) = F(t)$ and $F(1,t) = G(t)$. The following equation now holds:

$$\frac{F(1-u,t)}{1-u^2t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}\,.$$

Setting $u=1$ in this equation reduces it to (1.2).

A generalization of the Grigorchuk Formula in a completely different direction can be attempted: consider again a finitely generated group Γ, and an exact sequence

$$1 \longrightarrow \Xi \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow 1\,,$$

where this time Π is not necessarily free. Assume Π is generated as a monoid by a finite set S. Let again g_n be the number of words of length n in Π evaluating to 1 in Γ, and let f_n be the number of elements of Ξ whose minimal-length representation as a word in S has length n. Is there again a relation between the f_n and the g_n ? In Section 8 we derive such a relation when Π is the modular group $\mathbf{PSL}_2(\mathbf{Z})$.

Again there is a combinatorial counterpart; rather than considering graphs one considers a locally finite cellular complex $\mathcal{K}$ such that all vertices have isomorphic neighbourhoods. As before, g_n counts the number of paths of length n in the 1-skeleton of $\mathcal{K}$ between two fixed vertices; and f_n counts elements of the fundamental groupoid, i.e. homotopy classes of paths, between two fixed vertices whose minimal-length representation as a path in the 1-skeleton of $\mathcal{K}$ has length n. We obtain a relation between these numbers when $\mathcal{K}$ consists solely of triangles and arcs, with no two triangles nor two arcs meeting; these are precisely the complexes associated with quotients of the modular group.

The original motivation for our research was the study of cogrowth in group theory [Gri78a]; however, as it turned out, the more general problem in graph theory has applications to other domains of mathematics, like the Ihara-Selberg zeta function and its evaluation by Hyman Bass [Bas92].

2. Main result

Let $\mathcal{X}$ be a graph, that may have multiple edges and loops. We make the following typographical convention for the power series that will appear: a series in the formal variable t is written $G(t)$, or G for short, and $G(x)$ refers to the series G with x substituted for t. Functions are written on the right, with $(x)f$ or x^f denoting f evaluated at x.