Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Autor: Bartholdi, Laurent
Kapitel: 3.2 The series F and G on their circle of convergence
DOI: https://doi.org/10.5169/seals-64442

3.2 The series F and G on their circle of convergence

In this subsection we study the singularities the series F and G may have on their circle of convergence. The smallest positive real singularity has a special importance:

DEFINITION 3.4. For a series $f(t)$ with positive coefficients, let $\rho(f)$ denote its radius of convergence. Then f is $\rho(f)$*-recurrent* if

$$\lim_{t \to \rho(f)} f(t) = \infty \, .$$

Otherwise, it is $\rho(f)$*-transient.*

As typical examples, $1/(\rho - t)$ is ρ-recurrent, as are all rational series; $\sqrt{\rho - t}$ is ρ-transient, while $1/\sqrt{\rho - t}$ is not.

To study the singularities of F or G, we may suppose that $\star = \dagger$; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of F and G do not depend on the choice of $\star$ and $\dagger$. We make that assumption for the remainder of the subsection. We will also suppose throughout that $\mathcal{X}$ is d-regular, that the radius of convergence of F is $1/\alpha$ and the radius of convergence of G is $1/(d\nu) = 1/\beta$.

DEFINITION 3.5. Let $\mathcal{X}$ be a connected graph. A *proper cycle* in $\mathcal{X}$ is a proper circuit $(\pi_1, \dots, \pi_n)$ such that $\overline{\pi_1} \neq \pi_n$. The *proper period* p and *strong proper period* p_s are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\} \, ,$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists} \\ \text{a proper cycle } \pi \text{ in } \mathcal{B}(x,n) \text{ with } |\pi| = n\} \, ,$$

where by convention the gcd of the empty set is ∞. The graph $\mathcal{X}$ is *strongly properly periodic* if $p = p_s$.

The *period* q and *strong period* q_s of $\mathcal{X}$ are defined analogously with 'proper cycle' replaced by 'circuit'. $\mathcal{X}$ is *strongly periodic* if $q = q_s$.

THEOREM 3.6 (Cartwright [Car92]). *Let $\mathcal{X}$ have proper period p and strong proper period p_s. Then the singularities of F on its circle of convergence are among the*

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s \, .$$

If moreover $\mathcal{X}$ is strongly properly periodic, the singularities of F on its circle of convergence are precisely these numbers.

Let $\mathcal{X}$ have period q and strong period q_s. Then the singularities of G on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \dots, q_s .$$

If moreover $\mathcal{X}$ is strongly periodic, the singularities of G on its circle of convergence are precisely these numbers.

If $\mathcal{X}$ is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant N such that for all $x \in V(\mathcal{X})$ there is at x a circuit of odd length bounded by N, then $\mathcal{X}$ is strongly 1-periodic; otherwise $\mathcal{X}$ is strongly 2-periodic. The singularities of G on its circle of convergence are then at $1/\beta$, and also at $-1/\beta$ if $\mathcal{X}$ is strongly periodic with period 2.

If $\mathcal{X}$ is not strongly periodic, there may be one or two singularities on G's circle of convergence; consider for instance the 4-regular tree, and at a vertex $\star$ delete two or three edges replacing them by loops. The resulting graphs $\mathcal{X}_2$ and $\mathcal{X}_3$ are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$\begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned} \tag{3.3}$$

G_2 has singularities at $\pm 1/\sqrt{12}$ on its circle of convergence, while G_3 has only $2/7$ as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if $\beta < d$ the singularities of F on its circle of convergence are in bijection with those of G, so are at $1/\alpha$ and possibly $-1/\alpha$, if $\mathcal{X}$ is strongly two-periodic. If $\beta = d$, though, $\mathcal{X}$ can have any strong proper period; consider for example the cycles on length k studied in Section 7.2: they are strongly properly k-periodic.

The forthcoming simple result shows how $\mathcal{X}$ can be approximated by finite subgraphs.

LEMMA 3.7. *Let $\mathcal{X}$ be a graph and x, y two vertices in $\mathcal{X}$. Let $\mathfrak{G}_{x,y}$ and $\mathfrak{F}_{x,y}$ be the path series and enriched path series respectively from x to y in $\mathcal{X}$, and let $\mathfrak{G}^n_{x,y}$ and $\mathfrak{F}^n_{x,y}$ be the path series and enriched path series respectively from x to y in the ball $\mathcal{B}(x,n)$ (they are 0 if $y \notin \mathcal{B}(x,n)$). Then*

$$\lim_{n\to\infty} \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}\,, \qquad \lim_{n\to\infty} \mathfrak{F}^n_{x,y} = \mathfrak{F}_{x,y}\,.$$

Proof. Recall that $\lim \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}$ means that both terms are sums of paths, say A_n and A, such that the minimal length of paths in the symmetric difference $A_n \triangle A$ tends to infinity. Now the difference between $\mathfrak{G}^n_{x,y}$ and $\mathfrak{G}_{x,y}$ consists only of paths in $\mathcal{X}$ that exit $\mathcal{B}(x,n)$, and thus have length at least $2n - \delta(x,y) \to \infty$. The same argument holds for $\mathfrak{F}$. □

DEFINITION 3.8. The graph $\mathcal{X}$ is *quasi-transitive* if $\mathrm{Aut}(\mathcal{X})$ acts with finitely many orbits.

LEMMA 3.9. *Let $\mathcal{X}$ be a regular quasi-transitive connected graph with distinguished vertex $\star$, and let f_n and g_n denote respectively the number of proper circuits and circuits at $\star$ of length n. Then*

$$\limsup_{n\to\infty} g_n/\beta^n = \limsup_{n\to\infty} f_n/\alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

Proof. If $\mathcal{X}$ is finite, then $\beta = d$, the degree of $\mathcal{X}$; after a large even number of steps, a random walk starting at $\star$ will be uniformly distributed over $\mathcal{X}$ (or over the vertices at even distance of $\star$, in case all circuits have even length). A long enough walk then has probability $1/|\mathcal{X}|$ (or $2/|\mathcal{X}|$ if all circuits have even length) of being a circuit.

If $\mathcal{X}$ is infinite, we consider two cases. If $G(1/\beta) < \infty$, i.e. G is $1/\beta$-transient, the general term g_n/β^n of the series $G(1/\beta)$ tends to 0. If G is $1/\beta$-recurrent, then, as $\mathcal{X}$ is quasi-transitive, $\beta = d$ by [Woe98, Theorem 7.7]. We then approximate $\mathcal{X}$ by the sequence of its balls of radius R, by Lemma 3.7:

$$\lim_{n\to\infty} \frac{g_n}{\beta^n} = \lim_{R,n\to\infty} \frac{g_{R,n}}{d^n} = \lim_{R\to\infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star,R)|} = 0\,,$$

where we expand the circuit series of $\mathcal{B}(\star,R)$ as $\sum g_{R,n}t^n$.

The same proof holds for the f_n. Its particular case where $\mathcal{X}$ is a Cayley graph appears in [Woe83]. □

Note that if $\mathcal{X}$ is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if $\mathcal{X}$ is transient or null-recurrent then the common limsup is 0. If $\mathcal{X}$ is positive-recurrent then the limsups are normalized coefficients of $\mathcal{X}$'s Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary d-regular graphs: consider for instance the graph $\mathcal{X}_3$ described above. Its circuit series G_3, given in (3.3), has radius of convergence $1/\beta = 2/7$, and one easily checks that all its coefficients g_n satisfy $g_n/\beta^n \geq 1/2$.

We obtain the following characterization of rational series:

THEOREM 3.10. *For regular quasi-transitive connected graphs $\mathcal{X}$, the following are equivalent:*

1. *$\mathcal{X}$ is finite;*
2. *$G(t)$ is a rational function of t;*
3. *$F(t)$ is a rational function of t, and $\mathcal{X}$ is not an infinite tree.*

Proof. By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case $F(t) = 1$, Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that $F(t) = \sum f_n t^n$ is rational, not equal to 1. As the f_n are positive, F has a pole, of multiplicity m, at $1/\alpha$. There is then a constant $a > 0$ such that $f_n > a\binom{n}{m-1}\alpha^n$ for infinitely many values of n [GKP94, page 341]. It follows by Lemma 3.9 that $m = 1$ and the graph $\mathcal{X}$ is finite, of cardinality at most $1/a$. □

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

3.3 APPLICATION TO LANGUAGES

Let S be a finite set of cardinality d and let $\bar{\cdot}$ be an involution on S. A *word* is an element w of the free monoid S^*. A *language* is a set L of words. The language L is called *saturated* if for any $u, v \in S^*$ and $s \in S$ we have

$$uv \in L \Longleftrightarrow us\bar{s}v \in L;$$

that is to say, L is stable under insertion and deletion of subwords of the form $s\bar{s}$. The language L is called *desiccated* if no word in L contains a subword of the form $s\bar{s}$. Given a language L we may naturally construct its *saturation*