

## 6. Second proof of Theorem 2.4

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PROPOSITION 5.6 ([Ami80, Equation 4.4]). *Let  $X$  be a power series in  $t$  over a matrix ring, such that  $X(0) = \mathbf{1}$ . Then*

$$\det X = \exp\left(-\int \operatorname{tr}\left(\frac{\mathbf{1} - X}{Xt}\right) dt\right),$$

where the integration is the formal linear operation on power series that maps  $t^r$  to  $t^{r+1}/(r+1)$ .

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned} \frac{\det M}{(1 + (1 - u)t)^n (1 - (1 - u)^2 t^2)^m} &= \det \frac{M}{\mathbf{1} + (1 - u)Jt} \\ &= \exp\left(-\int \operatorname{tr} \frac{\mathbf{1} + (1 - u)Jt - M}{Mt} dt\right) \\ &= \exp\left(-\int \text{series counting non-trivial circuits,} \right. \\ &\quad \left. \text{length shifted down by one} dt\right) \\ &= \exp\left(-\int \operatorname{tr} \frac{(1 - (1 - u)^2 t^2)\mathbf{1} - P}{Pt} dt\right) \\ &= \det \frac{P}{1 - (1 - u)^2 t^2} = \frac{\det P}{(1 - (1 - u)^2 t^2)^{|V(X)|}}. \end{aligned}$$

## 6. SECOND PROOF OF THEOREM 2.4

Let  $P = [\star, \dagger]$  be the set of paths in  $\mathcal{X}$  from  $\star$  to  $\dagger$ . As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in  $\Pi = \mathbf{Z}[[P]]$ , the  $\mathbf{Z}$ -module of functions from the set of paths to  $\mathbf{Z}$ . We embed subsets of  $P$  in  $\Pi$  by mapping a subset to its characteristic function:

$$P \supset A \mapsto \chi_A, \quad \text{with } (\pi)\chi_A = \begin{cases} 1 & \text{if } \pi \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathcal{B}$  be the subset of bounded non-negative elements of  $\Pi$  (i.e. functions  $f$  such that there is a constant  $N$  with  $0 \leq (\pi)f < N$  for all paths  $\pi$ ). If  $\ell$  is a complete labelling of  $\mathcal{X}$ , there is an induced labelling  $\ell_*: \mathcal{B} \rightarrow \mathbf{k}$  given by

$$(f)\ell_* = \sum_{\pi \in P} (\pi)f \pi^\ell.$$

Note that the sum, although infinite, defines an element of  $\mathbf{k}$  due to the fact that  $\ell$  is complete.

DEFINITION 6.1 (Bump Scheme). Let  $e \in E(\mathcal{X})$  and  $v \in V(\mathcal{X})$ . A *squiggle along  $e$*  is a sequence  $(e, \bar{e}, \dots, e, \bar{e})$ . A *squiggle at  $v$*  is a squiggle along  $e$  for some edge  $e$  such that  $e^\alpha = v$ .

Let  $\pi = (v_0, e_1, \dots, e_n, v_n)$  be a path of length  $n$  in  $\mathcal{X}$ . A *bump scheme* for  $\pi$  is a pair  $B = ((\beta_0, \dots, \beta_n), (\gamma_1, \dots, \gamma_n))$ , with

- for all  $i \in \{0, \dots, n\}$ , a finite (possibly empty) sequence  $\beta_i = (\beta_{i,1}, \dots, \beta_{i,t_i})$  of squiggles at  $v_i$ ;
- for all  $i \in \{1, \dots, n\}$ , a squiggle  $\gamma_i$  along  $e_i$ .

The *weight*  $|B|$  of the bump scheme  $B$  is defined as

$$|B| = \sum_{i=0}^n \sum_{j=1}^{t_i} (|\beta_{i,j}| - 1) + \sum_{i=1}^n |\gamma_i|.$$

Given a path  $\pi$  and a bump scheme  $B = (\beta, \gamma)$  for  $\pi$ , we obtain a new path  $\pi \vee B \in P$ , by setting

$$\pi \vee B = \beta_{0,1} \cdot \dots \cdot \beta_{0,t_0} \gamma_1 e_1 \beta_{1,1} \cdot \dots \cdot \gamma_n e_n \beta_{n,1} \cdot \dots \cdot \beta_{n,t_n},$$

where the product denotes concatenation.

We now define a linear map  $\phi: \Pi \rightarrow \Pi[[u]]$  by setting, for  $f \in \Pi$  and  $\pi \in P$ ,

$$(\pi)((f)\phi) = \sum_{(\rho, B): \rho \vee B = \pi} (u - 1)^{|B|} (\rho) f,$$

where the sum ranges over all pairs  $(\rho, B)$  where  $\rho \in P$  and  $B$  is a bump scheme for  $\rho$  such that  $\rho \vee B = \pi$ . Note that the sum is finite because the edges of  $\rho$  and of  $B$  form subsets of those of  $\pi$ .

LEMMA 6.2. For any path  $\pi$  we have

$$(6.1) \quad (\pi)((\chi_P)\phi) = u^{\text{bc}(\pi)}.$$

*Proof.* Say  $\pi = (\pi_1, \dots, \pi_n)$  has  $m \geq 0$  bumps, at indices  $b_1, \dots, b_m$  so that  $\pi_{b_i} = \overline{\pi_{b_{i+1}}}$ . We will show that the evaluation at  $\pi$  of the left-hand side of (6.1) yields  $u^m$ .

We claim there is a bijection between the subsets  $C$  of  $\{1, \dots, m\}$  and the pairs  $(\rho_C, B_C)$  where  $\rho_C$  is a path and  $B_C$  is a bump scheme for  $\rho_C$  with  $\pi = \rho_C \vee B_C$ ; and further  $|B_C| = |C|$ .

First, take a  $\rho$  and a  $B = (\beta, \gamma)$  such that  $\rho \vee B = \pi$ . The path  $\rho \vee B$  is obtained by shuffling together the edges of  $\rho$  and  $B$ , and this partitions the

edges of  $\pi$  in two classes, namely (i) those coming from  $\rho$  and (ii) those coming from  $\beta$  and  $\gamma$ . Let  $C \subset \{1, \dots, m\}$  be the indices of the bumps  $b_i$  in  $\pi$  coming from  $B$ , i.e. such that  $\pi_{b_i}$  and  $\pi_{b_{i+1}}$  belong to the class (ii). One direction of the bijection is then  $(\rho, B) \mapsto C$ .

Conversely, given a subset  $C$  consider the set  $D = \{b_i \mid i \in C\}$ . Split it in maximal-length runs of consecutive integers  $D = D_1 \sqcup \dots \sqcup D_t$ . For all runs  $D_i$  do the following: to  $D_i = \{j, j+1, \dots, j+2k-1\}$  of even cardinality associate a squiggle  $\gamma_j$  of length  $2k$  along  $\pi_j$ ; to  $D_i = \{j, j+1, \dots, j+2k-2\}$  of odd cardinality associate a squiggle  $\beta_{j,l}$  of length  $2k$  at  $v_{j-1}$ ; then delete in  $\pi$  the edges  $\pi_j, \dots, \pi_{j+2k-1}$ . This process constructs a bump scheme  $B = (\beta, \gamma)$  while pruning edges of  $\pi$ , giving a path  $\gamma$  with  $\gamma \vee B = \pi$ . These two constructions are inverses, proving the claimed bijection.

It now follows that

$$(\pi)(\chi_P)\phi = \sum_{C \in \{1, \dots, m\}} (u-1)^{|B_C|} = \sum_{r=0}^m (u-1)^r \binom{m}{r} = u^m. \quad \square$$

Let  $\ell' : E(\mathcal{X}) \rightarrow \mathbf{k}[[u]]$  be defined by

$$e^{\ell'} = \frac{1}{1 - (e\bar{e})^\ell (1-u)^2} e^\ell K_{e^\omega}.$$

We prove Theorem 2.4 by noting that  $\mathfrak{G}(\ell) = (\chi_P)\ell_*$ , that  $\mathfrak{F}(\ell) = (\chi_P\phi)\ell_*$ , and that for any  $f \in \Pi$  we have  $(f\phi)\ell_* = K_*(f)\ell'_*$ . To prove this last equality, take a path  $\pi = (\pi_1, \dots, \pi_n)$  on vertices  $v_0, \dots, v_n$ . Then

$$(\chi_{\{\pi\}}\phi)\ell_* = \sum_B (u-1)^{|B|} (\pi \vee B)^\ell,$$

where the sum ranges over all bump schemes for  $\pi$ , and

$$K_*\pi^{\ell'} = K_{v_0} \frac{1}{1 - (u-1)^2 (\pi_1 \bar{\pi}_1)^\ell} \pi_1^\ell K_{v_1} \cdots \frac{1}{1 - (u-1)^2 (\pi_n \bar{\pi}_n)^\ell} \pi_n^\ell K_{v_n}.$$

It is clear these last two lines are equal; for the power series expansion of the  $K_{v_i}$  correspond to all the possible squiggle sequences  $\beta_i$  at  $v_i$ , and the power series expansion of the  $1/(1 - (u-1)^2 (\pi_i \bar{\pi}_i)^\ell)$  correspond to all possible squiggles  $\gamma_i$  along  $\pi_i$ .