7. Examples

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

7. EXAMPLES

We give here examples of regular graphs and when possible compute independently the series F and G. In some cases it will be easier to compute F, while in others it will be simpler to compute G first. In all cases, once one of F and G has been computed, the other one can be obtained using Corollary 2.6.

In all the examples the graphs are vertex transitive, so the choice of $\star$ is unimportant. To simplify the computations we choose $\dagger = \star$ and the length labelling.

7.1 COMPLETE GRAPHS

Let $\mathcal{X} = K_v$, the complete graph on $v \geq 3$ vertices. Its degree is $d = v-1$. To compute F and G, choose three distinct vertices $\star, \$, \#$ (the choice is unimportant as K_v is three-transitive). Define growth series

$F(u,t)$ the growth series of circuits based at $\star$;
$F'(u,t)$ the growth series of paths π from $\$$ to $\star$ with $\pi_1^\omega = \#$;
$F''(u,t)$ the growth series of paths π from $\$$ to $\star$ with $\pi_1^\omega = \star$.

Then

$$\begin{aligned} F &= 1 + (v-1)t\Big[(v-2)F' + uF''\Big], \\ F' &= t\Big[F'' + (v-3+u)F'\Big], \\ F'' &= t\Big[1 + (F-1)\frac{v-2+u}{v-1}\Big]. \end{aligned}$$

Indeed the first line states that a circuit at $\star$ is either the trivial circuit at $\star$, or a choice of one of $v-1$ edges to another point (call it $\$$), followed by a path from $\$$ to $\star$; this path can first go to any vertex of the $v-2$ vertices (say ($\#$)) different from $\star$ and $\$$, and thus contribute F', or go back to $\star$ and contribute F'' and a bump.

The second equation says that a path from $\$$ to $\star$ starting by going to $\#$ can either continue to $\star$, contributing F', go to any of the $v-3$ other vertices contributing F', or come back to $\$$, contributing F' and a bump.

The third line says that a path from $\$$ to $\star$ starting by going to $\star$ continues as a circuit at $\star$; but if the circuit is non-trivial, then one out of $v-1$ times a bump will be contributed.

Solving the system, we obtain

$$F(u,t) = \frac{1+(1-u)t}{1-(v-2+u)t} \cdot \frac{1-(v-2)t+(1-u)(v-2+u)t^2}{1+t+(1-u)(v-2+u)t^2} .$$

We then compute

$$G(t) = F(1,t) = \frac{1-(v-2)t}{(1+t)(1-(v-1)t)},$$

$$F(0,t) = \frac{(1+t)(1-(v-2)t+(v-2)t^2)}{(1-(v-2)t)(1+t+(v-2)t^2)} .$$

7.2 Cycles

Let $\mathcal{X} = C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0,t) = \frac{1+t^k}{1-t^k} .$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n = \sum_{i\in\mathbf{Z}\,:\, i\equiv 0\ [k],\ i\equiv n\ [2]} \binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t) = \frac{1}{k}\sum_{\zeta^k=1} \frac{1}{1-\left(\zeta+\zeta^{-1}\right)t} = \frac{1}{k}\sum_{j=0}^{k-1} \frac{1}{1-2\cos\left(\frac{2\pi j}{k}\right)t} .$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t) = \frac{(2t)^2+\left(1-\sqrt{1-4t^2}\right)^2}{(2t)^2-\left(1-\sqrt{1-4t^2}\right)^2} \cdot \frac{(2t)^k+\left(1-\sqrt{1-4t^2}\right)^k}{(2t)^k-\left(1-\sqrt{1-4t^2}\right)^k},$$

or, expanding,

$$G(t) = \frac{(2t)^k + \displaystyle\sum_{m=0}^{k/2}\left(1-4t^2\right)^m\binom{k}{2m}}{\displaystyle\sum_{m=1}^{(k+1)/2}\left(1-4t^2\right)^m\binom{k}{2m-1}} .$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u,t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.

For odd k, letting $k = 2\ell + 1$, we obtain

$$F(u,t) = \frac{1+(1-u)t}{1-(1+u)t} \cdot \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m (1+(1-u^2)t^2)^{\ell-m}}{\sum_{m=0}^{\ell} \alpha_m^\ell t^m (1+(1-u^2)t^2)^{\ell-m}},$$

$$G(t) = \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m}{(1-2t)\left(\sum_{m=0}^{\ell} \alpha_m^\ell t^m\right)},$$

where

$$\alpha_m^\ell = \begin{cases} (-)^{\frac{m}{2}} \binom{\ell - \frac{m}{2}}{\frac{m}{2}} & \text{if } m \equiv 0 \ [2], \\ (-)^{\frac{m-1}{2}} \binom{\ell - \frac{m+1}{2}}{\frac{m-1}{2}} & \text{if } m \equiv 1 \ [2]. \end{cases}$$

For even k, with $k = 2\ell$,

$$F(u,t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m (1-(1-u^2)t^2)^{\ell-2m}}{(1-(1+u)^2t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m (1-(1-u^2)t^2)^{\ell-1-2m}\right)},$$

$$G(t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m}{(1-4t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m\right)},$$

expressed as reduced fractions.

The first few values of F, where $\Box$ stands for $1+(1-u^2)t^2$, are:

k	$F(u,t)$	k	$F(u,t)$
1	$\dfrac{1+(1-u)t}{1-(1+u)t}$	2	$\dfrac{\Box}{1-(1+u)^2t^2}$
3	$\dfrac{(1+(1-u)t)(\Box-t)}{(1-(1+u)t)(\Box+t)}$	4	$\dfrac{\Box^2-2t^2}{1-(1+u)^2t^2}$
5	$\dfrac{(1+(1-u)t)(\Box^2-\Box t-t^2)}{(1-(1+u)t)(\Box^2+\Box t+t^2)}$	6	$\dfrac{\Box^2-3t^2}{(1-(1+u)^2t^2)(\Box^2-t^2)}$
7	$\dfrac{(1+(1-u)t)(\Box^3-\Box^2t-2\Box t^2+t^3)}{(1-(1+u)t)(\Box^3+\Box^2t-2\Box t^2-t^3)}$	8	$\dfrac{\Box^4-4\Box^2t^2+2t^4}{(1-(1+u)^2t^2)(\Box^2-2t^2)}$
9	$\dfrac{(1+(1-u)t)(\Box-t)(\Box^3-3\Box t^2-t^3)}{(1-(1+u)t)(\Box+t)(\Box^3-3\Box t^2+t^3)}$	10	$\dfrac{\Box^4-5\Box^2t^2+5t^4}{(1-(1+u)^2t^2)(\Box^4-3\Box^2t^2+t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*™.

7.3 Trees

Let $\mathcal{X}$ be the d-regular tree. Then

$$F(0,t)=1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u,t)$ and also obtain the answer using Corollary 2.6.

Let $\mathcal{T}$ be a regular tree of degree d with a fixed root $\star$, and let $\mathcal{T}'$ be the connected component of $\star$ in the two-tree forest obtained by deleting in $\mathcal{T}$ an edge at $\star$. Let $F(u,t)$ and $F'(u,t)$ respectively count circuits at $\star$ in $\mathcal{T}$ and $\mathcal{T}'$. For instance if $d=2$ then F' counts circuits in $\mathbf{N}$ and F counts circuits in $\mathbf{Z}$. For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F'. Then we have

$$F'=u+(d-1)\,tF't\frac{1}{1-(d-2+u)\,tF't},$$
$$F=1+dtF't\frac{1}{1-(d-1+u)\,tF't}.$$

Indeed a circuit in $\mathcal{T}'$ is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d-1$ directions, then a 'subcircuit' not returning to $\star$, then a step back to $\star$, followed by a step in any of $d-1$ directions (counting an extra factor of u if it was the same as before), a subcircuit, etc. If the 'subcircuit' is the empty circuit, it contributes a bump, hence the convention on F'. Likewise, a circuit in $\mathcal{T}$ is either the empty circuit (now counted as 1) or a sequence of circuits in subtrees each isomorphic to $\mathcal{T}'$.

We solve these equations to

$$F'(1-u,t) = \frac{2(1-u)}{1-u(d-u)t^2+\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}},$$

$$F(1-u,t) = \frac{2(d-1)(1-u^2t^2)}{(d-2)(1+u(d-u)t^2)+d\sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}}.$$

Using (2.3) and $F(0,t)=1$ we would obtain

$$G(t) = \frac{1+(d-1)\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2}{1-\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2},$$

or, after simplification,

$$G(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}},$$

which could have been obtained by setting $u=0$ in $F(1-u,t)$.

In particular if $d=2$, then $\mathcal{X}=C_\infty=\mathbf{Z}$ and

$$G(t) = \sum_{n\geq 0}\binom{2n}{n}t^{2n} = \frac{1}{\sqrt{1-4t^2}}.$$

Note that for all d the d-regular tree $\mathcal{X}$ is the Cayley graph of $\Gamma=(\mathbf{Z}/2\mathbf{Z})^{*d}$ with standard generating set. If d is even, $\mathcal{X}$ is also the Cayley graph of a free group of rank $d/2$ generated by a free set. We have thus computed the spectral radius of a random walk on a freely generated free group: it is, for $(\mathbf{Z}/2\mathbf{Z})^{*d}$ or for $\mathbf{F}_{d/2}$, equal to

$$\frac{2\sqrt{d-1}}{d}. \tag{7.1}$$

Remark that for $d=2$ the series $F(u,t)$ does have a simple series expansion. By direct expansion, we obtain the number of circuits of length $2n$ in $\mathbf{Z}$, with m local extrema, as

$$(t^{2n}u^m \mid F(u,t)) = \begin{cases} 2\binom{n-1}{\frac{m-1}{2}}^2 & \text{if } m\equiv 1\ [2],\\ 2\binom{n-1}{\frac{m}{2}}\binom{n-1}{\frac{m-2}{2}} & \text{if } m\equiv 0\ [2].\end{cases}$$

We may even look for a richer generating series than F: let

$$H(u,v,t) = \sum_{\pi\text{: path starting at } \star} u^{\mathrm{bc}(\pi)} v^{\delta(\star, \pi_{|\pi|})} t^{|\pi|} \in \mathbf{N}[u,v][[t]] ,$$

where δ denotes the graph distance. Then

$$\begin{aligned} H(1,v,t) &= F(1,t) + dF'tvF + dF'tv(d-1)F'tvF + \dots \\ &= \frac{1+F'(1,t)tv}{1-(d-1)F'(1,t)tv} F(1,t) ; \end{aligned}$$

and as H is a sum of series counting paths between fixed vertices we obtain $H(u,v,t)$ from $H(1,v,t)$ by extending (2.2) linearly:

$$\frac{H(1-u,v,t)}{1-u^2t^2} = \frac{H\left(1, v, \frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2} .$$

We could also have started by computing

$$H(0,v,t) = \frac{1+vt}{1-(d-1)vt} ,$$

the growth series of all proper paths in $\mathcal{T}$, and using (2.3) and (2.5) obtain

$$H(1,v,t) = \frac{1+\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2t}\right)^2}{1-u^2\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}\right)^2} \cdot H\left(\frac{1-\sqrt{1-4(d-1)t^2}}{2(d-1)t}, 0, v\right) ,$$

$$H(u,v,t) = \frac{1-t^2u^2}{1+u(d-u)t^2} \cdot \frac{(d-1)(4t^2+\square^2)}{4(d-1)^2t^2-u^2\square^2} \cdot \frac{2(d-1)t+v\square}{2t-v\square} ,$$

where $\square = 1+u(d-u)t^2 - \sqrt{(1+u(d-u)t^2)^2-4(d-1)t^2}$.

Recall that the growth series of a graph $\mathcal{X}$ at a base point $\star$ is the power series

$$P(t) = \sum_{v \in V(\mathcal{X})} t^{\delta(\star,v)} ,$$

where δ denotes the distance in $\mathcal{X}$. The series H is very general in that it contains a lot of information on $\mathcal{T}$, namely

- $H(u,0,t) = F(u,t)$;
- $H(0,1,t) = \frac{1+t}{1-(d-1)t} = P(t)$ is the growth series of $\mathcal{T}$;
- $H(1,1,t) = 1/(1-dt)$ is the growth series of all paths in $\mathcal{T}$.

(Note that these substitutions yield well-defined series because for any i there are only finitely many monomials having t-degree equal to i.)

We can also use this series H to compute the circuit series F_C of the cycle of length k, that was found in the previous section. Indeed the universal cover of a cycle is the regular tree $\mathcal{T}$ of degree 2, and circuits in C correspond bijectively to paths in $\mathcal{T}$ from $\star$ to any vertex at distance a multiple of k. We thus have

$$F_C(u,t) = \sum_{\zeta:\zeta^k=1} H(u,\zeta,t)$$

where the sum runs over all kth roots of unity and $d=2$ in H.

We consider next the following graphs: take a d-regular tree and fix a vertex $\star$. At $\star$, delete e vertices and replace them by e loops. Then clearly

$$F(0,t) = \frac{1+t}{1-(e-1)t},$$

as all the non-backtracking paths are constrained to the e loops. Using (2.3), we obtain after simplifications

$$G(t) = \frac{2(d-1)}{d+e-2-2e(d-1)t+(d-e)\sqrt{1-4(d-1)t^2}}. \tag{7.2}$$

The radius of convergence of G is

$$\min\left\{\frac{1}{2\sqrt{d-1}}, \frac{e-1}{d+e^2-2e}\right\}.$$

7.4 TOUGHER EXAMPLES

In this subsection we outline the computations of F and G for more complicated graphs. They are only provided as examples and are logically independent from the remainder of the paper. The arguments will therefore be somewhat condensed.

First take for $\mathcal{X}$ the Cayley graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}$ with generators $(0,-1) = `\downarrow'$, $(0,1) = `\uparrow'$ and $(1,0) = `\leftrightarrow'$. Geometrically, $\mathcal{X}$ is a doubly-infinite two-poled ladder.

In Subsection 7.3 we computed

$$F_{\mathbf{Z}}(u,t) = \frac{1-(1-u)^2t^2}{\sqrt{(1+(1-u^2)t^2)^2-4t^2}},$$

the growth of circuits restricted to one pole of the ladder. A circuit in $\mathcal{X}$ is a circuit in $\mathbf{Z}$, before and after each step ($\uparrow$ or $\downarrow$) of which we may switch to the other pole (with a $\leftrightarrow$) as many times as we wish, subject to the condition that the circuit finish at the same pole as it started. This last condition is expressed by the fact that the series we obtain must have only coefficients of even degree in t. Thus, letting $\mathrm{even}(f) = \frac{f(t)+f(-t)}{2}$, we have

$$G(t) = \text{even}\Big(\frac{1}{1-t}F_{\mathbf{Z}}\Big(1, \frac{t}{1-t}\Big)\Big);$$

it is then simple to obtain $F(u,t)$ by performing the substitution (2.3).

The following direct argument also gives $F(u,t)$: a walk on the ladder is obtained from a walk on a pole (i.e. on $\mathbf{Z}$) by inserting before and after every step on a pole a (possibly empty) sequence of steps from one pole to the other. This process is expressed by performing on $F_{\mathbf{Z}}$ the substitution

$$t \mapsto t + t^2 + t^3u + t^4u^2 + \cdots = t + \frac{t^2}{1-tu},$$

corresponding to replacing a step on a pole by itself, or itself followed by a step to the other pole, or itself, a step to the other pole and a step back, etc. But if the path had a bump at the place the substitution was performed, this bump would disappear when a step is added from one pole to the other. In formulas,

$$tu \mapsto tu + t^2 + t^3u + t^4u^2 + \cdots = tu + \frac{t^2}{1-tu}.$$

Finally we must add at the beginning of the path a sequence of steps from one pole to the other. Therefore we obtain

$$F(u,t) = \text{even}\Big\{\Big(1 + \frac{t}{1-tu}\Big)F_{\mathbf{Z}}\Big(\frac{tu + t^2/(1-tu)}{t + t^2/(1-tu)}, t + \frac{t^2}{1-tu}\Big)\Big\}.$$

As another example, consider the group $\mathbf{Z}$ generated by the non-free set $\{\pm1, \pm2\}$. Geometrically, it can be seen as the set of points $(2i, 0)$ and $(2i+1, \sqrt{3})$ for all $i \in \mathbf{Z}$, with edges between all points at Euclidean distance 2 apart; but we will not make use of this description. The circuit series of $\mathbf{Z}$ with this enlarged generating set will be an algebraic function of degree 4 over the rationals.

Define first the following series:

$f(t)$ counts the walks from 0 to 0 in $\mathbf{N}$;

$g(t)$ counts the walks from 0 to 1 in $\mathbf{N}$;

$h(t)$ counts the walks from 1 to 1 in $\mathbf{N}$.

Denote the generators of $\mathbf{Z}$ by $1 = \uparrow$, $2 = \uparrow\uparrow$, $-1 = \downarrow$ and $-2 = \downarrow\downarrow$. The series then satisfy the following equations, where the generators' symbol is written instead of 't' to make the formulas self-explanatory:

$$f = 1 + (\uparrow f \downarrow + \uparrow g \downarrow\downarrow + \uparrow\uparrow g \downarrow + \uparrow\uparrow h \downarrow\downarrow)f,$$
$$g = f \uparrow f + f \uparrow\uparrow g,$$
$$h = f + f \downarrow g + g \downarrow\downarrow g,$$

giving a solution f that is algebraic of degree 4 over $\mathbf{Z}(t)$.

Then define the following series:

G counts the walks from 0 to 0 in $\mathbf{Z}$;

e counts the walks from 0 to 1 in $\mathbf{Z}$.

They satisfy the equations

$$G = 1 + 2(\uparrow f \downarrow G + \upuparrows g \downarrow G + \uparrow f \downdownarrows e + \upuparrows g \downdownarrows e + \uparrow g \downdownarrows G + \upuparrows h \downdownarrows G),$$
$$e = G \uparrow f + G \upuparrows f + G \upuparrows g$$

giving the solution

$$G = \frac{4 + 3t - 6t^2 - 10t(1+2t)\delta + 2t^2(3+8t)\delta^2 - 6t^4(1+t)\delta^3}{4 - 7t - 36t^2}$$

where δ is a root of the equation

$$1 - (2t+1)\delta + t(2+3t)\delta^2 - t^2(1+2t)\delta^3 + t^4\delta^4 = 0\,.$$

8. Cogrowth of non-free presentations

We perform here a computation extending the results of Section 3.1. The general setting, expressed in the language of group theory, is the following: let Π be a group generated by a finite set S and let $\Xi < \Pi$ be any subgroup. We consider the following generating series:

$$F(t) = \sum_{\gamma \in \Xi < \Pi} t^{|\gamma|},$$
$$G(t) = \sum_{\substack{\text{words } w \text{ in } S \\ \text{defining an element in } \Xi}} t^{|w|},$$

where $|\gamma|$ is the minimal length of γ in the generators S, and $|w|$ is the usual length of the word w. Is there some relation between these series? In case Π is quasi-free on S, the relation between F and G is given by Corollary 2.6. We consider two other examples: Π quasi-free but on a set smaller than S, and $\Pi = \mathbf{PSL}_2(\mathbf{Z})$.