7.1 COMPLETE GRAPHS

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7. Examples

We give here examples of regular graphs and when possible compute independently the series F and G. In some cases it will be easier to compute F, while in others it will be simpler to compute G first. In all cases, once one of F and G has been computed, the other one can be obtained using Corollary 2.6.

In all the examples the graphs are vertex transitive, so the choice of $\star$ is unimportant. To simplify the computations we choose $\dagger = \star$ and the length labelling.

7.1 Complete graphs

Let $\mathcal{X} = K_v$, the complete graph on $v \geq 3$ vertices. Its degree is $d = v-1$. To compute F and G, choose three distinct vertices $\star, \$, \#$ (the choice is unimportant as K_v is three-transitive). Define growth series

$F(u,t)$ the growth series of circuits based at $\star$;
$F'(u,t)$ the growth series of paths π from \$ to $\star$ with $\pi_1^\omega = \#$;
$F''(u,t)$ the growth series of paths π from \$ to $\star$ with $\pi_1^\omega = \star$.

Then

$$\begin{aligned}
F &= 1 + (v-1)t\Big[(v-2)F' + uF''\Big],\\
F' &= t\Big[F'' + (v-3+u)F'\Big],\\
F'' &= t\Big[1 + (F-1)\frac{v-2+u}{v-1}\Big].
\end{aligned}$$

Indeed the first line states that a circuit at $\star$ is either the trivial circuit at $\star$, or a choice of one of $v-1$ edges to another point (call it \$), followed by a path from \$ to $\star$; this path can first go to any vertex of the $v-2$ vertices (say (#)) different from $\star$ and \$, and thus contribute F', or go back to $\star$ and contribute F'' and a bump.

The second equation says that a path from \$ to $\star$ starting by going to # can either continue to $\star$, contributing F', go to any of the $v-3$ other vertices contributing F', or come back to \$, contributing F' and a bump.

The third line says that a path from \$ to $\star$ starting by going to $\star$ continues as a circuit at $\star$; but if the circuit is non-trivial, then one out of $v-1$ times a bump will be contributed.

Solving the system, we obtain

$$F(u,t)=\frac{1+(1-u)t}{1-(v-2+u)t}\cdot\frac{1-(v-2)t+(1-u)(v-2+u)t^2}{1+t+(1-u)(v-2+u)t^2}.$$

We then compute

$$G(t)=F(1,t)=\frac{1-(v-2)t}{(1+t)(1-(v-1)t)},$$
$$F(0,t)=\frac{(1+t)(1-(v-2)t+(v-2)t^2)}{(1-(v-2)t)(1+t+(v-2)t^2)}.$$

7.2 Cycles

Let $\mathcal{X}=C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0,t)=\frac{1+t^k}{1-t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n=\sum_{i\in\mathbf{Z}:\,i\equiv 0\ [k],\ i\equiv n\ [2]}\binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t)=\frac{1}{k}\sum_{\zeta^k=1}\frac{1}{1-\left(\zeta+\zeta^{-1}\right)t}=\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{1-2\cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t)=\frac{(2t)^2+\left(1-\sqrt{1-4t^2}\right)^2}{(2t)^2-\left(1-\sqrt{1-4t^2}\right)^2}\cdot\frac{(2t)^k+\left(1-\sqrt{1-4t^2}\right)^k}{(2t)^k-\left(1-\sqrt{1-4t^2}\right)^k},$$

or, expanding,

$$G(t)=\frac{(2t)^k+\sum_{m=0}^{k/2}\left(1-4t^2\right)^m\binom{k}{2m}}{\sum_{m=1}^{(k+1)/2}\left(1-4t^2\right)^m\binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u,t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.