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Solving the system, we obtain

$$F(u,t) = \frac{1+(1-u)t}{1-(v-2+u)t} \cdot \frac{1-(v-2)t+(1-u)(v-2+u)t^2}{1+t+(1-u)(v-2+u)t^2} .$$

We then compute

$$G(t) = F(1,t) = \frac{1-(v-2)t}{(1+t)(1-(v-1)t)},$$

$$F(0,t) = \frac{(1+t)(1-(v-2)t+(v-2)t^2)}{(1-(v-2)t)(1+t+(v-2)t^2)} .$$

7.2 Cycles

Let $\mathcal{X} = C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0,t) = \frac{1+t^k}{1-t^k} .$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n = \sum_{i\in\mathbf{Z}:\, i\equiv 0\ [k],\ i\equiv n\ [2]} \binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t) = \frac{1}{k}\sum_{\zeta^k=1} \frac{1}{1-\left(\zeta+\zeta^{-1}\right)t} = \frac{1}{k}\sum_{j=0}^{k-1} \frac{1}{1-2\cos\left(\frac{2\pi j}{k}\right)t} .$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t) = \frac{(2t)^2+\left(1-\sqrt{1-4t^2}\right)^2}{(2t)^2-\left(1-\sqrt{1-4t^2}\right)^2} \cdot \frac{(2t)^k+\left(1-\sqrt{1-4t^2}\right)^k}{(2t)^k-\left(1-\sqrt{1-4t^2}\right)^k},$$

or, expanding,

$$G(t) = \frac{(2t)^k + \sum_{m=0}^{k/2}\left(1-4t^2\right)^m \binom{k}{2m}}{\sum_{m=1}^{(k+1)/2}\left(1-4t^2\right)^m \binom{k}{2m-1}} .$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u,t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.

For odd k, letting $k = 2\ell + 1$, we obtain

$$F(u,t) = \frac{1 + (1-u)t}{1-(1+u)t} \cdot \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m (1 + (1-u^2)t^2)^{\ell-m}}{\sum_{m=0}^{\ell} \alpha_m^\ell t^m (1 + (1-u^2)t^2)^{\ell-m}},$$

$$G(t) = \frac{\sum_{m=0}^{\ell} \alpha_m^\ell (-t)^m}{(1-2t)\left(\sum_{m=0}^{\ell} \alpha_m^\ell t^m\right)},$$

where

$$\alpha_m^\ell = \begin{cases} (-)^{\frac{m}{2}} \binom{\ell - \frac{m}{2}}{\frac{m}{2}} & \text{if } m \equiv 0\ [2], \\ (-)^{\frac{m-1}{2}} \binom{\ell - \frac{m+1}{2}}{\frac{m-1}{2}} & \text{if } m \equiv 1\ [2]. \end{cases}$$

For even k, with $k = 2\ell$,

$$F(u,t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m (1 - (1-u^2)t^2)^{\ell - 2m}}{(1 - (1+u)^2 t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell - 1 - m}{m} (-t^2)^m (1 - (1-u^2)t^2)^{\ell - 1 - 2m}\right)},$$

$$G(t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell - m} \binom{\ell - m}{m} (-t^2)^m}{(1 - 4t^2)\left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell - 1 - m}{m} (-t^2)^m\right)},$$

expressed as reduced fractions.

The first few values of F, where $\square$ stands for $1 + (1 - u^2)t^2$, are:

k	$F(u,t)$	k	$F(u,t)$
1	$\frac{1+(1-u)t}{1-(1+u)t}$	2	$\frac{\square}{1-(1+u)^2t^2}$
3	$\frac{(1+(1-u)t)(\square - t)}{(1-(1+u)t)(\square + t)}$	4	$\frac{\square^2 - 2t^2}{1-(1+u)^2t^2}$
5	$\frac{(1+(1-u)t)(\square^2 - \square t - t^2)}{(1-(1+u)t)(\square^2 + \square t + t^2)}$	6	$\frac{\square^2 - 3t^2}{(1-(1+u)^2t^2)(\square^2 - t^2)}$
7	$\frac{(1+(1-u)t)(\square^3 - \square^2 t - 2\square t^2 + t^3)}{(1-(1+u)t)(\square^3 + \square^2 t - 2\square t^2 - t^3)}$	8	$\frac{\square^4 - 4\square^2 t^2 + 2t^4}{(1-(1+u)^2t^2)(\square^2 - 2t^2)}$
9	$\frac{(1+(1-u)t)(\square - t)(\square^3 - 3\square t^2 - t^3)}{(1-(1+u)t)(\square + t)(\square^3 - 3\square t^2 + t^3)}$	10	$\frac{\square^4 - 5\square^2 t^2 + 5t^4}{(1-(1+u)^2t^2)(\square^4 - 3\square^2 t^2 + t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*™.

7.3 Trees

Let $\mathcal{X}$ be the d-regular tree. Then

$$F(0,t) = 1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u,t)$ and also obtain the answer using Corollary 2.6.

Let $\mathcal{T}$ be a regular tree of degree d with a fixed root $\star$, and let $\mathcal{T}'$ be the connected component of $\star$ in the two-tree forest obtained by deleting in $\mathcal{T}$ an edge at $\star$. Let $F(u,t)$ and $F'(u,t)$ respectively count circuits at $\star$ in $\mathcal{T}$ and $\mathcal{T}'$. For instance if $d = 2$ then F' counts circuits in $\mathbf{N}$ and F counts circuits in $\mathbf{Z}$. For a reason that will become clear below, we make the convention that the empty circuit is counted as '1' in F and as 'u' in F'. Then we have

$$F' = u + (d-1)\,tF't\frac{1}{1-(d-2+u)\,tF't},$$
$$F = 1 + dtF't\frac{1}{1-(d-1+u)\,tF't}.$$

Indeed a circuit in $\mathcal{T}'$ is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d-1$ directions, then