Zeitschrift: L'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Kapitel: 9. Free products of graphs
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DOI: https://doi.org/10.5169/seals-64442

THEOREM 8.5. *With the notation introduced above, let* Ξ *be any subgroup of* Π *not intersecting* $\{t^\gamma \mid t \in T, \gamma \in \Pi\}$ *and let* $F(t)$, $G(t)$ *be the "cogrowth" series and circuit series of* $\Xi\backslash\Pi$. *Let* $\mathcal{E}$ *be the complete graph on* $|\Upsilon|$ *vertices and let* $G_{\mathcal{E}}(t)$, $G_{\mathcal{E}}^{\neq}(t)$ *count the circuits and the non-closing paths respectively in* $\mathcal{E}$. *Define the function* ϕ *by*

$$\left(\frac{t^2}{1+(|S|-1)t^4}\right)\phi = \frac{tG_{\mathcal{E}}}{1+(G_{\mathcal{E}}-G_{\mathcal{E}}^{\neq})((|S|-1)G_{\mathcal{E}}+G_{\mathcal{E}}^{\neq})t^2}\,.$$

Then we have

$$F(t) \sim G((t)\phi)\,.$$

9. FREE PRODUCTS OF GRAPHS

We give here a general construction combining two pointed graphs and show how to compute the generating functions for circuits in the "product" in terms of the generating functions for circuits in the factors.

DEFINITION 9.1 (Free Product, [Que94, Definition 4.8]). Let $(\mathcal{E},\star)$ and $(\mathcal{F},\star)$ be two connected pointed graphs. Their *free product* $\mathcal{E}*\mathcal{F}$ is the graph constructed as follows: start with copies of $\mathcal{E}$ and $\mathcal{F}$ identified at $\star$; at each vertex v apart from $\star$ in $\mathcal{E}$, respectively $\mathcal{F}$, glue a copy of $\mathcal{F}$, respectively $\mathcal{E}$, by identifying v and the $\star$ of the copy. Repeat the process, each time glueing $\mathcal{E}$'s and $\mathcal{F}$'s to the new vertices.

If (E,S), (F,T) are two groups with fixed generators whose Cayley graphs are $\mathcal{E}$ and $\mathcal{F}$ respectively, then $\mathcal{E}*\mathcal{F}$ is the Cayley graph of $(E*F, S\sqcup T)$.

We now compute the circuit series of $\mathcal{E}*\mathcal{F}$ in terms of the circuit series of $\mathcal{E}$ and $\mathcal{F}$. Let $G_{\mathcal{E}}$, $G_{\mathcal{F}}$ and $G_{\mathcal{X}}$ be the generating functions counting circuits in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{X}=\mathcal{E}*\mathcal{F}$ respectively. We will use the following description: given a circuit at $\star$ in $\mathcal{X}$, it can be decomposed as a product of circuits never passing through $\star$. Each of these circuits, in turn, starts either in the $\mathcal{E}$ or the $\mathcal{F}$ copy at $\star$. Say one starts in $\mathcal{E}$; it can then be expressed as a circuit in $\mathcal{E}$ never passing through $\star$, and such that at all vertices, except the first and last, a circuit starting in $\mathcal{F}$ has been inserted. Moreover, any choice of such circuits satisfying these conditions will give a circuit at $\star$ in $\mathcal{X}$, and different choices will yield different circuits.

Let $H_{\mathcal{E}}$ (respectively $H_{\mathcal{F}}$) be the generating function counting non-trivial circuits in $\mathcal{E}$ (respectively $\mathcal{F}$) never passing through $\star$. Obviously

$$G_{\mathcal{E}} = \frac{1}{1 - H_{\mathcal{E}}} \quad \text{so} \quad H_{\mathcal{E}} = 1 - \frac{1}{G_{\mathcal{E}}}.$$

Let $L_{\mathcal{E}}$ (respectively $L_{\mathcal{F}}$) be the generating function counting non-trivial circuits in $\mathcal{X}$ never passing through $\star$ and starting in $\mathcal{E}$ (respectively $\mathcal{F}$). Then

$$L_{\mathcal{E}}(t) = H_{\mathcal{E}}\Big(\frac{t}{1 - L_{\mathcal{F}}(t)}\Big) \cdot (1 - L_{\mathcal{F}}(t)),$$
$$L_{\mathcal{F}}(t) = H_{\mathcal{F}}\Big(\frac{t}{1 - L_{\mathcal{E}}(t)}\Big) \cdot (1 - L_{\mathcal{E}}(t)).$$

Indeed write $H_{\mathcal{E}} = \sum h_n t^n$. Then by the description given above

$$L_{\mathcal{E}} = \sum h_n t^n \Big(\frac{1}{1 - L_{\mathcal{F}}}\Big)^{n-1},$$

which is precisely the given formula. Finally

$$G_{\mathcal{X}} = \frac{1}{1 - L_{\mathcal{E}} - L_{\mathcal{F}}}.$$

Writing $M_{\mathcal{E}} = t/(1 - L_{\mathcal{E}})$ and $M_{\mathcal{F}} = t/(1 - L_{\mathcal{F}})$, we simplify these equations to

$$1 - \frac{t}{M_{\mathcal{E}}} = \Big(1 - \frac{1}{G_{\mathcal{E}}(M_{\mathcal{F}})}\Big) \cdot \frac{t}{M_{\mathcal{F}}},$$
$$1 - \frac{t}{M_{\mathcal{F}}} = \Big(1 - \frac{1}{G_{\mathcal{F}}(M_{\mathcal{E}})}\Big) \cdot \frac{t}{M_{\mathcal{E}}},$$
$$G_{\mathcal{X}} = \frac{1}{1 - (1 - \frac{t}{M_{\mathcal{E}}}) - (1 - \frac{t}{M_{\mathcal{F}}})} = \frac{1/t}{1/M_{\mathcal{E}} + 1/M_{\mathcal{F}} - 1/t},$$

so

$$\text{(9.1)} \qquad \frac{1}{M_{\mathcal{E}}} + \frac{1}{M_{\mathcal{F}}} - \frac{1}{t} = \frac{1}{M_{\mathcal{E}} G_{\mathcal{F}}(M_{\mathcal{E}})} = \frac{1}{M_{\mathcal{F}} G_{\mathcal{E}}(M_{\mathcal{F}})} = \frac{1}{t G_{\mathcal{X}}}.$$

If f is a power series with $f(0) = 0$ and $f'(0) \neq 0$, let us write f^{-1} for the inverse of f; i.e. for the series g with $g(f(t)) = f(g(t)) = t$ (for instance, $f(t) = t$ is equal to its inverse; the inverse of $\frac{at+b}{ct+d}$ is $\frac{dt-b}{-ct+a}$).

From (9.1) we obtain $M_{\mathcal{E}} = (tG_{\mathcal{F}})^{-1} \circ (tG_{\mathcal{X}})$ and $M_{\mathcal{F}} = (tG_{\mathcal{E}})^{-1} \circ (tG_{\mathcal{X}})$; so composing (9.1) with $(tG_{\mathcal{X}})^{-1}$ we obtain the

THEOREM 9.2.

$$\frac{1}{(tG_{\mathcal{X}})^{-1}} = \frac{1}{(tG_{\mathcal{E}})^{-1}} + \frac{1}{(tG_{\mathcal{F}})^{-1}} - \frac{1}{t}\,. \tag{9.2}$$

An equation equivalent to this one, though not trivially so, appeared in a paper by Gregory Quenell [Que94], and, in yet another language, in a paper by Dan Voiculescu [Voi90, Theorem 4.5].

We can use (9.2) to obtain by a different method the circuit series of regular trees (see Section 7.3). Indeed the free product of regular trees of degree d and e is a regular tree of degree $d+e$. Letting G_d denote the circuit series of a regular tree of degree d, we "guess" that

$$G_d(t) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}}\,,$$

and verify that the limit

$$G_1(t) = \lim_{d\to 1} \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)t^2}} = \frac{1}{1-t^2}$$

is indeed the circuit series of the 1-regular tree. Then we compute

$$(tG_d)^{-1}(u) = \frac{2u}{2-d+d\sqrt{1+4u^2}}\,;$$

if we define $\triangle_d$ by

$$\triangle_d := \frac{1}{(tG_d)^{-1}(u)} - \frac{1}{u} = d\frac{\sqrt{1+4u^2}-1}{2u}\,,$$

it satisfies $\triangle_d + \triangle_e = \triangle_{d+e}$ and we have proved that our guess of G_d is correct for all $d \geq 1$, in light of (9.2).

As another application of (9.2), we compute the circuit series $G(t)$ of the Cayley graph of $\mathbf{PSL}_2(\mathbf{Z}) = \langle a, b \mid a^2, b^3 \rangle$ with generators $\{a, b, b^{-1}\}$. This graph is the free product of the 1-regular tree $\mathcal{E}$ and of the 3-cycle $\mathcal{F}$. We know from Section 7.1 that

$$G_{\mathcal{E}} = \frac{1}{1-t^2}\,, \qquad G_{\mathcal{F}} = \frac{1-t}{(1+t)(1-2t)}$$

are the circuit series of $\mathcal{E}$ and $\mathcal{F}$. We then compute

$$(tG_{\mathcal{E}})^{-1}(u) = \frac{\sqrt{1+4u^2}-1}{2u}\,,$$

$$(tG_{\mathcal{F}})^{-1}(u) = \frac{1+u-\sqrt{1-2u+9u^2}}{2(1-2u)}\,,$$

so after some lucky simplifications

$$G(t) = \frac{1}{t}\Big(\frac{1}{1/(tF_{\mathcal{E}})^{-1}(u) + 1/(tG_{\mathcal{F}})^{-1}(u) - 1/u}\Big)^{-1}$$
$$= \frac{(2-t)\sqrt{1-2t-5t^2+6t^3+t^4} - t + t^2 + t^3}{2(1-2t-5t^2+6t^3)} .$$

(A closed form such as this one does not exist for $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/k\mathbf{Z})$ with $k > 3$, because then the series G is algebraic of degree greater than 2.)

COROLLARY 9.3. *If the circuit series of $\mathcal{E}$ and $\mathcal{F}$ are both algebraic, then the circuit series of $\mathcal{E} * \mathcal{F}$ is also algebraic.*

Proof. Sums and products of algebraic series are algebraic. If f satisfies the algebraic relation $P(f, t) = 0$, then its formal inverse satisfies the relation $P(t, f^{-1}) = 0$ so is also algebraic. □

Recall the notions of radius of convergence and ρ-recurrence given in Definition 3.4.

LEMMA 9.4. *We have*

$$\rho(f) = \sup_t (tf(t))^{-1} ,$$

where the supremum is taken over all t such that the series $(tf)^{-1}$ converges. If f is ρ-recurrent, then also

$$\rho(f) = \lim_{t\to\infty} (tf(t))^{-1} .$$

Proof. Clearly $\rho(f) = \rho(tf)$; if tf converges over $[0, \rho[$ then $(tf)^{-1}$ converges over $[0, \sigma[$ where $\sigma = \rho f(\rho)$; then we have $\lim_{t\to\sigma}(tf)^{-1} = \rho$. The second assertion follows because in this case $\sigma = \infty$. □

COROLLARY 9.5. *Let the circuit series of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{X} = \mathcal{E} * \mathcal{F}$ be $G_{\mathcal{E}}$, $G_{\mathcal{F}}$ and $G_{\mathcal{X}}$ respectively, and suppose all three series are recurrent. Then*

$$1/\rho(G_{\mathcal{X}}) = 1/\rho(G_{\mathcal{E}}) + 1/\rho(G_{\mathcal{F}}) .$$

Proof. This follows from

$$\begin{aligned} 1/\rho(G_{\mathcal{X}}) &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{X}})^{-1}} \\ &= \lim_{t\to\infty} \frac{1}{(tG_{\mathcal{E}})^{-1}} + \frac{1}{(tG_{\mathcal{F}})^{-1}} - \frac{1}{t} \qquad \text{by (9.2)} \\ &= 1/\rho(G_{\mathcal{E}}) + 1/\rho(G_{\mathcal{F}}) - 0\,. \qquad \square \end{aligned}$$

Note that the corollary does not extend to non-recurrent series; for instance, it fails if $\mathcal{E} = \mathcal{F} = \mathbf{Z}$. Indeed then

$$G_{\mathcal{E}} = G_{\mathcal{F}} = \frac{1}{\sqrt{1-4t^2}}\,, \qquad \rho(G_{\mathcal{E}}) = \rho(G_{\mathcal{F}}) = 1/4\,,$$
$$G_{\mathcal{X}} = \frac{3}{1+2\sqrt{1-12t^2}}\,, \qquad \rho(G_{\mathcal{X}}) = 1/\sqrt{12}\,.$$

10. Direct products of graphs

There are two natural definitions for *direct products* of graphs; they correspond to direct products of groups with generating set either the union or cartesian product of the generating sets of the factors. A general treatment of products of graphs can be found in [CDS79, pages 65 and 203].

Definition 10.1. If S is a set, the *stationing graph* on S is the graph $\mathcal{X} = \Sigma_S$ with $V(\mathcal{X}) = E(\mathcal{X}) = S$, where for the edges $s^\alpha = s^\omega = \bar{s} = s$ hold.

Lemma 10.2. *Let $\mathcal{X}$ be a graph, and $\mathcal{E} = \mathcal{X} \sqcup \Sigma_{\mathcal{X}}$ be the graph obtained by adding a loop to every vertex in $\mathcal{X}$. Let $G_{\mathcal{X}}$ and $G_{\mathcal{E}}$ be the growth functions for circuits in $\mathcal{X}$ and $\mathcal{E}$ respectively. Then we have*

$$G_{\mathcal{E}}(t) = \frac{1}{1-t} G_{\mathcal{X}}\left(\frac{t}{1-t}\right).$$

Definition 10.3 (First Product). Let $\mathcal{E}$ and $\mathcal{F}$ be two graphs. Their *direct product* $\mathcal{X} = \mathcal{E} \times \mathcal{F}$ is defined by

$$V(\mathcal{X}) = V(\mathcal{E}) \times V(\mathcal{F})$$

and

$$E(\mathcal{X}) = (E(\mathcal{E}) \times \Sigma_{\mathcal{F}}) \sqcup (\Sigma_{\mathcal{E}} \times E(\mathcal{F}))\,.$$