

Zeitschrift: L'Enseignement Mathématique
Band: 45 (1999)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COUNTING PATHS IN GRAPHS
Kapitel: 11. FURTHER WORK
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DOI: <https://doi.org/10.5169/seals-64442>

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11. FURTHER WORK

It was mentioned in Subsection 3.3 how the main result applies to languages. This is a special case of a much more general problem:

PROBLEM 11.1. *Given a language L and a set \mathcal{U} of words, define the desiccation $L_{\mathcal{U}}$ of L as the set of words in L containing no $u \in \mathcal{U}$ as a subword.*

Give sufficient conditions on L and \mathcal{U} such that a formula exist relating $\Theta(L)$ and $\Theta(L_{\mathcal{U}})$.

The special case we studied in this paper is that of

$$\mathcal{U} = \{s\bar{s} \mid s \in S\}$$

and a sufficient condition is that L be saturated.

For general \mathcal{U} this is not always sufficient: let $S = \{a, b\}$ and $L = b^*(ab^*ab^*)^*$ be the set of words with an even number of a 's. Then if $\mathcal{U} = \{a^2\}$ there are 7 desiccated words of length 5:

$$\{b^5, ab^3a, ab^2ab, abab^2, bab^2a, babab, b^2aba\}$$

and if $\mathcal{U}' = \{b^2\}$ there are 6 desiccated words of length 5:

$$\{babab, ba^4, aba^3, a^2ba^2, a^3ba, a^4b\}.$$

The growth series of \mathcal{U} and \mathcal{U}' are the same, namely t^2 , but the growth series of $L_{\mathcal{U}}$ and $L_{\mathcal{U}'}$ differ in their degree-5 coefficient.

We gave in Section 9 a formula relating the circuit series of a free product to the circuit series of its factors. There is a notion of *amalgamated product* of graphs, that is a direct analogue of the amalgamated product of groups.

PROBLEM 11.2. *What conditions on $\mathcal{D}, \mathcal{E}, \mathcal{F}$ are sufficient so that*

$$\frac{1}{(zG_{\mathcal{X}})^{-1}} = \frac{1}{(zG_{\mathcal{E}})^{-1}} + \frac{1}{(zG_{\mathcal{F}})^{-1}} - \frac{1}{(zG_{\mathcal{D}})^{-1}}$$

*where $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is an amalgamated product of \mathcal{E} and \mathcal{F} along \mathcal{D} ?*

The formula holds if \mathcal{D} is the trivial graph; but it cannot hold in general. If $\mathcal{E} = \mathcal{F}$ is the "ladder graph" described in Section 7.4: the set of points (i, j) with $i \in \mathbf{Z}$ and $j \in \{0, 1\}$, with edges connecting all pairs of vertices at Euclidean distance 1, and \mathcal{D} is \mathbf{Z} , embedded as a pole of the ladder, then

the amalgamated product $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is isomorphic to \mathbf{Z}^2 . The circuit series of \mathcal{D} , \mathcal{E} and \mathcal{F} have been calculated explicitly and are algebraic. The circuit series of \mathcal{X} was shown in Section 10 to be transcendental; so there can exist no algebraic definition of $G_{\mathcal{X}}$ in terms of $G_{\mathcal{D}}$, $G_{\mathcal{E}}$ and $G_{\mathcal{F}}$. However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph \mathcal{X} , one can construct a graph $\mathcal{X}^{(k)}$ on the same vertex set, and with edge set the set of paths of length $\leq k$ in \mathcal{X} . Is there some simple relation between the path series of \mathcal{X} and of $\mathcal{X}^{(k)}$? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's R -transform [Voi90]. His S -transform, in terms of graphs, corresponds to $\mathcal{E} * \mathcal{F}$ with as edge set all sequences (e, f) and (f, e) , for $e \in E(\mathcal{E})$ and $f \in E(\mathcal{F})$. Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

12. ACKNOWLEDGEMENTS

The main result of this paper was found in Rome thanks to the nurturing of Tullio and Katiuscia Ceccherini-Silberstein and their family, whom I thank. Many people heard or read preliminary often obscure versions and provided valuable feedback; I am grateful to (in order of appearance) Michel Kervaire, Shalom Eliahou, Pierre de la Harpe, Fabrice Liardet, Rostislav Grigorchuk, Alain Valette, Étienne Ghys, Igor Lysionok, Jean-Paul Allouche, Gilles Robert, Thierry Vust, Robyn Curtis, Hung Fioramonti, and Vaughan Jones for their help and interest.

Added in proof. Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at <http://www.math.berkeley.edu/~vfr/plnalg1.ps>).