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11. Further work

It was mentioned in Subsection 3.3 how the main result applies to languages. This a special case of a much more general problem:

PROBLEM 11.1. *Given a language L and a set $\mathcal{U}$ of words, define the desiccation $L_{\mathcal{U}}$ of L as the set of words in L containing no $u \in \mathcal{U}$ as a subword.*

Give sufficient conditions on L and $\mathcal{U}$ such that a formula exist relating $\Theta(L)$ and $\Theta(L_{\mathcal{U}})$.

The special case we studied in this paper is that of

$$\mathcal{U} = \{s\bar{s} \mid s \in S\}$$

and a sufficient condition is that L be saturated.

For general $\mathcal{U}$ this is not always sufficient: let $S = \{a, b\}$ and $L = b^*(ab^*ab^*)^*$ be the set of words with an even number of a's. Then if $\mathcal{U} = \{a^2\}$ there are 7 desiccated words of length 5 :

$$\{b^5, ab^3a, ab^2ab, abab^2, bab^2a, babab, b^2aba\}$$

and if $\mathcal{U}' = \{b^2\}$ there are 6 desiccated words of length 5 :

$$\{babab, ba^4, aba^3, a^2ba^2, a^3ba, a^4b\}.$$

The growth series of $\mathcal{U}$ and $\mathcal{U}'$ are the same, namely t^2, but the growth series of $L_{\mathcal{U}}$ and $L_{\mathcal{U}'}$ differ in their degree-5 coefficient.

We gave in Section 9 a formula relating the circuit series of a free product to the circuit series of its factors. There is a notion of *amalgamated product* of graphs, that is a direct analogue of the amalgamated product of groups.

PROBLEM 11.2. *What conditions on $\mathcal{D}, \mathcal{E}, \mathcal{F}$ are sufficient so that*

$$\frac{1}{(zG_{\mathcal{X}})^{-1}} = \frac{1}{(zG_{\mathcal{E}})^{-1}} + \frac{1}{(zG_{\mathcal{F}})^{-1}} - \frac{1}{(zG_{\mathcal{D}})^{-1}}$$

where $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is an amalgamated product of $\mathcal{E}$ and $\mathcal{F}$ along $\mathcal{D}$?

The formula holds if $\mathcal{D}$ is the trivial graph; but it cannot hold in general. If $\mathcal{E} = \mathcal{F}$ is the "ladder graph" described in Section 7.4: the set of points (i,j) with $i \in \mathbf{Z}$ and $j \in \{0, 1\}$, with edges connecting all pairs of vertices at Euclidean distance 1, and $\mathcal{D}$ is $\mathbf{Z}$, embedded as a pole of the ladder, then the amalgamated product $\mathcal{X} = \mathcal{E} *_{\mathcal{D}} \mathcal{F}$ is isomorphic to $\mathbf{Z}^2$. The circuit series of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ have been calculated explicitly and are algebraic. The circuit series of $\mathcal{X}$ was shown in Section 10 to be transcendental; so there can exist no algebraic definition of $G_{\mathcal{X}}$ in terms of $G_{\mathcal{D}}$, $G_{\mathcal{E}}$ and $G_{\mathcal{F}}$. However, there exists some relations between these series, as given by [Voi90, Theorem 5.5].

Given a graph $\mathcal{X}$, one can construct a graph $\mathcal{X}^{(k)}$ on the same vertex set, and with edge set the set of paths of length $\leq k$ in $\mathcal{X}$. Is there some simple relation between the path series of $\mathcal{X}$ and of $\mathcal{X}^{(k)}$? This could be useful for example to obtain asymptotics about the cogrowth of a group subject to enlargement of generating set [Cha93].

The equation (9.2) corresponds to Voiculescu's R-transform [Voi90]. His S-transform, in terms of graphs, corresponds to $\mathcal{E} * \mathcal{F}$ with as edge set all sequences (e,f) and (f,e), for $e \in E(\mathcal{E})$ and $f \in E(\mathcal{F})$. Is there an analogue to Theorem 9.2 in this context?

Finally, (9.2) computes the circuit series of a free product in terms of the circuit series of the factors. A more complicated formula yields the path series of a free product in terms of the path series of the factors. Such considerations give another derivation of the results in Section 8.

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Added in proof. Recently Vaughan Jones has obtained very similar results in the context of planar algebras, for which some 'path' and 'proper path' series give the Hilbert-Poincaré series of a planar algebra over different subalgebras (see *Planar Algebras I*; preprint at http://www.math.berkeley.edu/~vfr/plnalg1.ps).