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**THEOREM 4.** *Let  $\Gamma$  be a Fuchsian group without elliptic elements (an element  $\gamma \in \text{PSL}(2, \mathbf{R})$  is elliptic if  $|\text{tr}(\gamma)| < 2$  where  $\text{tr}$  is the trace). Then  $\mathbf{H}/\Gamma$  is a complete connected orientable Riemannian manifold of dimension 2 with a metric of constant curvature  $-1$ .*

**DEFINITION.** *A hyperbolic surface is a connected orientable manifold  $M = \mathbf{H}/\Gamma$  as in Theorem 4 (where  $\Gamma$  is a Fuchsian group without elliptic elements).  $M$  is called closed if  $M$  is compact and has no boundary.*

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

**DEFINITION** (Compare Figure 2). Let  $g \geq 2$  be an integer. A *canonical polygon*  $P(g)$  is a polygon with  $4g$  sides, denoted by  $a_1, \dots, a_{4g}$ , ordered clockwise, and angles  $\alpha_i$  between  $a_i$  and  $a_{i+1}$ ,  $i = 1, \dots, 4g$  (indices are taken modulo  $4g$ ), such that

- (I)  $a_i$  and  $a_{i+2g}$  have the same length,  $i = 1, \dots, 2g$ ;
- (II) the sum of the angles of  $P(g)$  is  $2\pi$ ;
- (III)  $0 < \alpha_i < \pi$ ,  $i = 1, \dots, 4g$ ;
- (IV)  $\alpha_1 = \alpha_{2g+1}$ ;
- (V)  $\sum_{i=1}^g \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^g \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$ .

I shall speak of condition (I) (or (II) or (III) or (IV) or (V) ) referring to this definition.

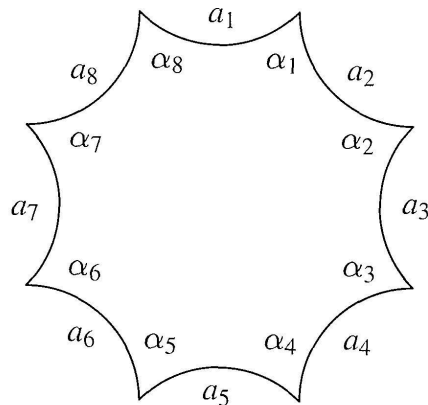


FIGURE 2  
A canonical polygon  $P(g)$  for  $g = 2$

REMARKS. (i) Note that, by condition (II), both sides of the equation in condition (V) equal  $\pi$ .

(ii) The terminology *canonical* polygon is not standard, one finds different objects called canonical polygons in the literature (see for example in [15]).

DEFINITION. Let  $\Gamma$  be a Fuchsian group. A *fundamental domain* for  $\Gamma$  is a measurable subset  $D$  of  $\mathbf{H}$  such that

- (i)  $\bigcup_{\gamma \in \Gamma} \gamma(D) = \mathbf{H}$ , and
- (ii)  $\text{int}(\bar{D}) \cap \text{int}(\gamma(\bar{D})) = \emptyset$  for  $id \neq \gamma \in \Gamma$ . Here,  $\text{int}(S)$  is the *interior* of a set  $S$  and  $id$  is the unit matrix.

THEOREM 5 (Poincaré). A *canonical polygon*  $P = P(g)$  is the *fundamental domain* of a Fuchsian group  $\Gamma$  and  $\mathbf{H}/\Gamma$  is a closed hyperbolic surface of genus  $g$ . The group  $\Gamma$  is generated by the  $2g$  elements  $\gamma_i$  where  $\gamma_i$  is defined by the conditions  $\gamma_i(P) \cap \text{int}(P) = \emptyset$  and  $\gamma_i(a_i) = a_{i+2g}$  if  $i$  is odd and  $\gamma_i(a_{i+2g}) = a_i$  if  $i$  is even,  $i = 1, \dots, 2g$ .

REMARKS. (i) For a proof see for example Poincaré [10], Siegel [15], Beardon [1], Iversen [5]. The theorem holds for much more general polygons. A general proof was first given by Maskit [9] and by de Rham [11].

(ii) Traditionally, the  $2g$  generators  $\gamma_i$  of a Fuchsian group corresponding to a closed hyperbolic surface of genus  $g$  are chosen such that the relation

$$\prod_{i=1}^{2g} [\gamma_{2i-1}, \gamma_{2i}] = id$$

holds where

$$[\gamma_{2i-1}, \gamma_{2i}] = \gamma_{2i-1} \gamma_{2i} (\gamma_{2i-1})^{-1} (\gamma_{2i})^{-1}.$$

With the choice made here, the relation

$$\gamma_1 \gamma_2 \cdots \gamma_{2g} (\gamma_1)^{-1} (\gamma_2)^{-1} \cdots (\gamma_{2g})^{-1} = id$$

holds. Compare the introduction for the reasons for this choice.

(iii) Let  $P(g)$  be a canonical polygon and  $M = \mathbf{H}/\Gamma$  be the corresponding closed hyperbolic surface. Then the vertices of  $P(g)$  correspond to a unique point  $Q$  in  $M$  and the side  $a_i$  (as well as  $a_{2g+i}$ ) of  $P(g)$  corresponds to a simple closed curve  $u_i$  in  $M$ ,  $i = 1, \dots, 2g$ . These curves all intersect transversally in  $Q$  and intersect in no other point. Moreover, these curves are geodesic loops based in  $Q$ , this means that the curves may have an angle  $\neq \pi$  in  $Q$ , but outside  $Q$ , they are geodesic. Further, condition (IV) and

condition (V) of canonical polygons are equivalent to the condition that  $u_1$  and  $u_2$  are simple closed geodesics in  $M$ .

#### 4. TRIGONOMETRY

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

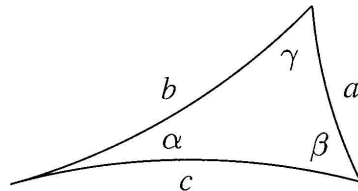


FIGURE 3

The notation for a triangle

THEOREM 6. Let  $T$  be a triangle with angles  $\alpha, \beta, \gamma$  and sides of length  $a, b, c$  with the notation of Figure 3. Then

- (i)  $\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$  ;
- (ii)  $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$  ;
- (iii)  $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$  .

LEMMA 7. Let  $T$  be a triangle with the notation of Figure 3. Let  $T'$  be a triangle with sides of length  $a', b', c'$  and angles  $\alpha', \beta', \gamma'$ . Let  $a = a'$  and  $b = b'$ . Then

$$c' > c \iff \gamma' > \gamma \iff \alpha' + \beta' < \alpha + \beta .$$

*Proof.* The first equivalence is a consequence of Theorem 6 (ii).

Let  $Z$  be the centre of the side  $c$  and let  $u$  be the geodesic segment, of length  $d/2$  say, between  $Z$  and the vertex  $C$  of  $T$ . The segment  $u$  separates  $T$  into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$