Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

Autor: SCHMUTZ SCHALLER, Paul

Kapitel: 3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DOI: https://doi.org/10.5169/seals-64444

THEOREM 4. *Let* Γ *be a Fuchsian group without elliptic elements (an element* $\gamma \in \mathrm{PSL}(2,\mathbf{R})$ *is elliptic if* $|\mathrm{tr}(\gamma)| < 2$ *where* tr *is the trace). Then* $\mathbf{H}/\Gamma$ *is a complete connected orientable Riemannian manifold of dimension* 2 *with a metric of constant curvature* -1.

DEFINITION. A *hyperbolic surface* is a connected orientable manifold $M = \mathbf{H}/\Gamma$ as in Theorem 4 (where Γ is a Fuchsian group without elliptic elements). M is called *closed* if M is compact and has no boundary.

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DEFINITION (Compare Figure 2). Let $g \geq 2$ be an integer. A *canonical polygon* $P(g)$ is a polygon with $4g$ sides, denoted by $a_1, \dots, a_{4g}$, ordered clockwise, and angles α_i between a_i and a_{i+1}, $i = 1, \dots, 4g$ (indices are taken modulo $4g$), such that

(I) a_i and a_{i+2g} have the same length, $i = 1, \dots, 2g$;

(II) the sum of the angles of $P(g)$ is 2π;

(III) $0 < \alpha_i < \pi$, $i = 1, \dots, 4g$;

(IV) $\alpha_1 = \alpha_{2g+1}$;

(V) $\displaystyle\sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$.

I shall speak of condition (I) (or (II) or (III) or (IV) or (V)) referring to this definition.

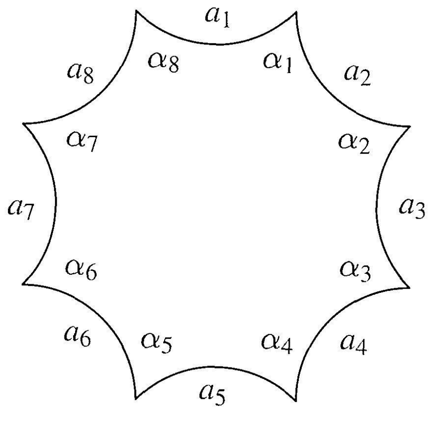

FIGURE 2

A canonical polygon $P(g)$ for $g = 2$

REMARKS. (i) Note that, by condition (II), both sides of the equation in condition (V) equal π.

(ii) The terminology *canonical* polygon is not standard, one finds different objects called canonical polygons in the literature (see for example in [15]).

DEFINITION. Let Γ be a Fuchsian group. A *fundamental domain* for Γ is a measurable subset D of $\mathbf{H}$ such that

(i) $\bigcup_{\gamma\in\Gamma}\gamma(D)=\mathbf{H}$, and

(ii) $int\left(\bar{D}\right)\cap int\left(\gamma\left(\bar{D}\right)\right)=\varnothing$ for $id\neq\gamma\in\Gamma$. Here, $int(S)$ is the *interior* of a set S and id is the unit matrix.

THEOREM 5 (Poincaré). *A canonical polygon $P=P(g)$ is the fundamental domain of a Fuchsian group Γ and $\mathbf{H}/\Gamma$ is a closed hyperbolic surface of genus g. The group Γ is generated by the $2g$ elements γ_i where γ_i is defined by the conditions $\gamma_i(P)\cap int(P)=\varnothing$ and $\gamma_i(a_i)=a_{i+2g}$ if i is odd and $\gamma_i(a_{i+2g})=a_i$ if i is even, $i=1,\dots,2g$.*

REMARKS. (i) For a proof see for example Poincaré [10], Siegel [15], Beardon [1], Iversen [5]. The theorem holds for much more general polygons. A general proof was first given by Maskit [9] and by de Rham [11].

(ii) Traditionally, the $2g$ generators γ_i of a Fuchsian group corresponding to a closed hyperbolic surface of genus g are chosen such that the relation

$$\prod_{i=1}^{2g}\left[\gamma_{2i-1},\gamma_{2i}\right]=id$$

holds where

$$\left[\gamma_{2i-1},\gamma_{2i}\right]=\gamma_{2i-1}\gamma_{2i}(\gamma_{2i-1})^{-1}(\gamma_{2i})^{-1}.$$

With the choice made here, the relation

$$\gamma_1\gamma_2\dots\gamma_{2g}(\gamma_1)^{-1}(\gamma_2)^{-1}\dots(\gamma_{2g})^{-1}=id$$

holds. Compare the introduction for the reasons for this choice.

(iii) Let $P(g)$ be a canonical polygon and $M=\mathbf{H}/\Gamma$ be the corresponding closed hyperbolic surface. Then the vertices of $P(g)$ correspond to a unique point Q in M and the side a_i (as well as a_{2g+i}) of $P(g)$ corresponds to a simple closed curve u_i in M, $i=1,\dots,2g$. These curves all intersect transversally in Q and intersect in no other point. Moreover, these curves are geodesic loops based in Q, this means that the curves may have an angle $\neq\pi$ in Q, but outside Q, they are geodesic. Further, condition (IV) and condition (V) of canonical polygons are equivalent to the condition that u_1 and u_2 are simple closed geodesics in M.

4. TRIGONOMETRY

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

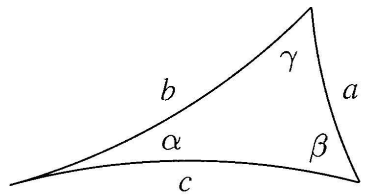

FIGURE 3
The notation for a triangle

THEOREM 6. *Let T be a triangle with angles α, β, γ and sides of length a, b, c with the the notation of Figure 3. Then*

(i) $\dfrac{\sinh a}{\sin \alpha} = \dfrac{\sinh b}{\sin \beta} = \dfrac{\sinh c}{\sin \gamma}$;

(ii) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$;

(iii) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$.

LEMMA 7. *Let T be a triangle with the notation of Figure 3. Let T' be a triangle with sides of length a', b', c' and angles α', β', γ'. Let $a = a'$ and $b = b'$. Then*

$$c' > c \Longleftrightarrow \gamma' > \gamma \Longleftrightarrow \alpha' + \beta' < \alpha + \beta .$$

Proof. The first equivalence is a consequence of Theorem 6 (ii).

Let Z be the centre of the side c and let u be the geodesic segment, of length $d/2$ say, between Z and the vertex C of T. The segment u separates T into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$