Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 45 (1999)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TEICHMÜLLER SPACE AND FUNDAMENTAL DOMAINS OF FUCHSIAN GROUPS

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Kapitel: 4. Trigonometry

DOI: https://doi.org/10.5169/seals-64444

condition (V) of canonical polygons are equivalent to the condition that u_1 and u_2 are simple closed geodesics in M.

4. TRIGONOMETRY

REMARK. By abuse of notation a side of a polygon will often be identified with its length.

The following theorem is standard (for a proof see for example [1], [2]).

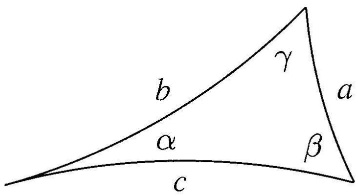

FIGURE 3
The notation for a triangle

THEOREM 6. *Let T be a triangle with angles α, β, γ and sides of length a, b, c with the the notation of Figure 3. Then*

(i) $\dfrac{\sinh a}{\sin \alpha} = \dfrac{\sinh b}{\sin \beta} = \dfrac{\sinh c}{\sin \gamma}$;

(ii) $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$;

(iii) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh c$.

LEMMA 7. *Let T be a triangle with the notation of Figure 3. Let T' be a triangle with sides of length a', b', c' and angles α', β', γ'. Let $a = a'$ and $b = b'$. Then*

$$c' > c \Longleftrightarrow \gamma' > \gamma \Longleftrightarrow \alpha' + \beta' < \alpha + \beta .$$

Proof. The first equivalence is a consequence of Theorem 6 (ii).

Let Z be the centre of the side c and let u be the geodesic segment, of length $d/2$ say, between Z and the vertex C of T. The segment u separates T into two triangles (compare Figure 4). Applying Theorem 6 (ii) to them, we obtain

$$\cosh a = \cosh(c/2) \cosh(d/2) - \sinh(c/2) \sinh(d/2) \cos \delta$$

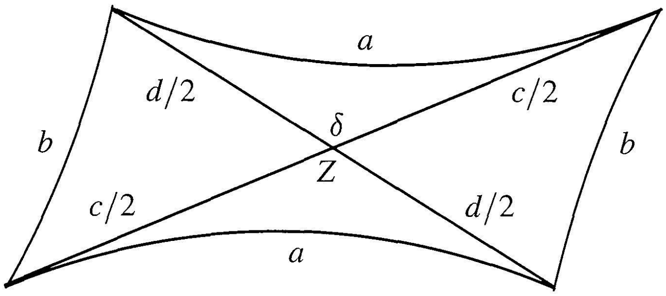

FIGURE 4
The triangle T (thick lines) is half of this quadrilateral

and

$$\cosh b = \cosh(c/2)\cosh(d/2) + \sinh(c/2)\sinh(d/2)\cos\delta$$

for an angle δ. This implies

$$\cosh a + \cosh b = 2\cosh(c/2)\cosh(d/2)\,. \tag{1}$$

Let $\widetilde{T}$ be the triangle with sides of length a, b, d (compare Figure 4). Then the angles of $\widetilde{T}$ are $\alpha+\beta, \gamma_1, \gamma_2$ with $\gamma = \gamma_1 + \gamma_2$. Now if the length of c grows, then the length of d diminishes (by (1)), therefore, applying the first equivalence of the lemma to the triangle $\widetilde{T}$, the angle $\alpha+\beta$ diminishes and the second equivalence of the lemma follows. □

COROLLARY 8. *Let Q and Q' be two quadrilaterals with the same lengths of the four sides. Let $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ be the four angles in Q and Q', respectively, in the natural order (α and γ are opposite). Then*

$$\alpha + \gamma > \alpha' + \gamma' \iff \beta + \delta < \beta' + \delta'\,.$$

Proof. Clear by Lemma 7 (draw a diagonal in Q and in Q'). □

LEMMA 9. *Let T be a triangle with the notation of Figure 3. Let $T(t)$ be a triangle with sides of length ta, tb, tc and angles $\alpha_t, \beta_t, \gamma_t$.*

(i) *If $t > 1$, then $\alpha_t < \alpha$, $\beta_t < \beta$, $\gamma_t < \gamma$.*

(ii) *For $t \to \infty$, the three angles $\alpha_t, \beta_t, \gamma_t$ converge to zero.*

Proof. (i) I prove $\gamma_t < \gamma$, the two other inequalities follow analogously. By Theorem 6 (ii) it has to be shown that

$$\frac{\cosh ta \cosh tb - \cosh tc}{\sinh ta \sinh tb} - \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b} > 0\,. \tag{2}$$

By symmetry we can assume that $a \geq b$. Consider the left hand side of (2) as a function $f = f(c)$ of c with fixed a, b, t. A calculation yields

$$f(a+b) = f(a-b) = 0. \tag{3}$$

Further, $f'(c) = 0$ implies

$$\frac{t \sinh tc}{\sinh c} = \frac{\sinh ta \sinh tb}{\sinh a \sinh b}$$

and by the convexity of the function sinh we conclude that $f'(c)$ has only one zero. Since $t > 1$, it follows (by the definition of f) that

$$f(c) \to -\infty \text{ for } c \to \pm\infty.$$

Therefore, by (3), $f(c) > 0$ for $a - b < c < a + b$, which is the triangle inequality, and $\gamma_t < \gamma$ follows.

(ii) Assume without restriction that $a \leq b \leq c$. It then follows by Theorem 6 (i) that $\alpha \leq \beta \leq \gamma$. This implies by Theorem 6 (iii) that α_t and β_t converge to zero for $t \to \infty$. We compare the triangle $T(t)$ with the triangle $T'(t)$ which has two sides of length $t(a+b)/2$ and one side of length tc. Denote by γ_t' the angle in $T'(t)$ which is opposite to the side of length tc. By a similar (but easier) argument as in part (i) it follows that $\gamma_t' \geq \gamma_t$ for all $t \geq 1$. It is therefore sufficient to prove

$$\gamma_t' \to 0, \quad \text{for } t \to \infty. \tag{4}$$

By Theorem 6 (i) we have

$$\sin \frac{\gamma_t'}{2} = \frac{\sinh(tc/2)}{\sinh(t(a+b)/2)}.$$

This implies (4) since $c/2 < (a+b)/2$ (by the triangle inequality). □

COROLLARY 10. *Let Q be a quadrilateral with sides of length a, b, c, d and angles $\alpha, \beta, \gamma, \delta$ (so that a and c are opposite sides and α and γ are opposite angles). Let $Q(t)$ be a quadrilateral with sides of length ta, tb, tc, td and angles $\alpha_t, \beta_t, \gamma_t, \delta_t$ (the notation is analogous to that of Q).*

(i) *If $t > 1$, then at least two opposite angles are smaller in $Q(t)$ than in Q.*

(ii) *For every $\epsilon > 0$, there exists a real $T(\epsilon)$ such that, for every $t > T(\epsilon)$, $\alpha_t + \gamma_t < \epsilon$ or $\beta_t + \delta_t < \epsilon$.*

Proof. Let e be the length of a diagonal of Q. Construct the quadrilateral $Q'(t)$ with a diagonal of length te and sides of length ta, tb, tc, td. By Lemma 9 all four angles of $Q'(t)$ are smaller than the corresponding angles in Q and moreover converge to zero if $t \to \infty$. The corollary now follows by Corollary 8. □