5. The geometric K-theory for π_0G finite

5. THE GEOMETRIC K-THEORY FOR $\pi_0 G$ FINITE

In this section we shall determine the geometric group $K^*(X,G)$ whenever G has only a finite number of connected components. The main point is the existence of a final object (namely $H\backslash G$, where H is the maximal compact subgroup of G) in the category of proper G-manifolds.

Throughout this section G is a Lie group with a finite number of connected components. H denotes the maximal compact subgroup of G. And $\mathfrak{g}$, $\mathfrak{h}$ are the Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{h}\backslash\mathfrak{g} \to 0.$$

Passing to dual spaces (over $\mathbf{R}$):

$$0 \leftarrow \mathfrak{h}^* \leftarrow \mathfrak{g}^* \leftarrow (\mathfrak{h}\backslash\mathfrak{g})^* \leftarrow 0.$$

By the co-adjoint representation H acts on $(\mathfrak{h}\backslash\mathfrak{g})^*$

$$(\mathfrak{h}\backslash\mathfrak{g})^* \times H \to (\mathfrak{h}\backslash\mathfrak{g})^*.$$

Given a G-manifold X, let H act on $X \times (\mathfrak{h}\backslash\mathfrak{g})^*$ by

$$(x,u)\,h = (xh, uh)$$

$(x \in X,\ u \in (\mathfrak{h}\backslash\mathfrak{g})^*,\ h \in H)$.

PROPOSITION 1. *For any G-manifold X there is a canonical isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h}\backslash\mathfrak{g})^*) \to K^i(X,G) \qquad (i = 0,1).$$

REMARK 2. The isomorphism of the proposition is completely canonical and has no shift of dimension.

COROLLARY 3. *Set* $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$. *If the co-adjoint action of H on* $(\mathfrak{h}\backslash\mathfrak{g})^*$ *is* Spin^c, *then*

$$K_H^i(X) \cong K^{i+\epsilon}(X,G).$$

Proof of Corollary 3. If the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c, then the Thom isomorphism [1] applies to give an isomorphism

$$K_H^i(X) \to K_H^{i+\epsilon}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*).$$

Composing this with the isomorphism of Proposition 1 proves the corollary. □

REMARK 4. Set $H\backslash G = \{Hg \mid g \in G\}$. There is the evident (right) action of G on $H\backslash G$

$$(H\backslash G) \times G \to H\backslash G.$$

The action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c if and only if $H\backslash G$ admits a G-invariant Spin^c-structure.

To analyze the case when the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is not Spin^c, fix an H-invariant Euclidean structure on $(\mathfrak{h}\backslash\mathfrak{g})^*$. Proceed as in [15]. Since H is connected, the co-adjoint representation maps H into $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$. Let $\mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^*$ be the non-trivial 2-fold covering of $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$ and form the commutative diagram

$$\begin{array}{ccc} \widetilde{H} & \longrightarrow & \mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^* \\ \downarrow & & \downarrow \\ H & \longrightarrow & \mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^* \end{array}$$

where $\widetilde{H} = H \times_{\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*} \mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^*$ is the 2-fold covering of H obtained by pulling-back the Spin covering of $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$. There is then ([1]) the Thom isomorphism

$$K^i_{\widetilde{H}}(X) \to K^{i+\epsilon}_{\widetilde{H}}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*).$$

Moreover, let $u \in \widetilde{H}$ be the non-identity element of $\widetilde{H}$ which maps to the identity element of H by the projection $\widetilde{H} \to H$. If E is any $\widetilde{H}$-vector bundle on X, there is the direct sum decomposition

$$E = E_+ \oplus E_-$$

where $E_\pm = \{v \in E \mid vu = \pm v\}$. This leads to a direct sum decomposition of $K^*_{\widetilde{H}}(X)$:

$$K^i_{\widetilde{H}}(X) = \left[K^i_{\widetilde{H}}(X)\right]_+ \oplus \left[K^i_{\widetilde{H}}(X)\right]_-,$$

where $\left[K^i_{\widetilde{H}}(X)\right]_\pm$ is obtained by only using $E_\pm$. Note that $\left[K^i_{\widetilde{H}}(X)\right]_+ \cong K^i_H(X)$.

COROLLARY 5. *For any G-manifold X, there is an isomorphism of abelian groups*

$$\left[K^i_{\widetilde{H}}(X)\right]_- \to K^{i+\epsilon}(X, G),$$

$i = 0, 1$, $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$.

Proof. The Thom isomorphism

$$K^i_{\widetilde{H}}(X) \to K^{i+\epsilon}_{\widetilde{H}}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*)$$

gives an isomorphism

$$\left[K^i_{\tilde{H}}(X)\right]_- \to K_H^{i+\epsilon}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*)\,.$$

Combining this with the isomorphism of Proposition 1 proves Corollary 5. □

The essential point in the proof of Proposition 1 is given by

LEMMA 6. *Let Z be any proper G-manifold. Then there exists a G-map from Z to $H\backslash G$.*

Proof. Assume for simplicity that $H\backslash G$ admits a G-invariant Riemannian metric of non-positive curvature. This is the case if G is semi-simple [17].

It follows easily from the slice theorem of Palais [23] that Z can be covered by open sets $U_0, U_1, U_2, \ldots$ such that each U_i is mapped into itself by G, $\{U_i\}$ is a locally finite cover of Z, and there exist G-maps $f_i\colon U_i \to H\backslash G$. Two points in $H\backslash G$ are joined by a unique geodesic. Let $\phi_0\colon U_0 \cup U_1 \to \mathbf{R}$, $\phi_1\colon U_0 \cup U_1 \to \mathbf{R}$ be a C^∞ partition of unity on $U_0 \cup U_1$ subordinate to the covering U_0, U_1 and with each ϕ_i constant on orbits. Then $\phi_0 f_0 + \phi_1 f_1$ is a G-map from $U_0 \cup U_1$ to $H\backslash G$ where $(\phi_0 f_0 + \phi_1 f_1)$ means the weighted average (by weights $\phi_0(x)$, $\phi_1(x)$) of $f_0(x), f_1(x)$ along the unique geodesic joining $f_0(x)$ and $f_1(x)$. Iterating this construction produces the desired G-map from Z to $H\backslash G$.

The general case has been proved by A. Borel [10]. □

Proof of Proposition 1. Let (Z, ξ, f) be a K-cocycle for (X, G). According to Lemma 6 there is a G-map $\theta\colon Z \to H\backslash G$. Let $h\colon Z \to X \times (H\backslash G)$ be

$$h(z) = (f\,z, \theta z)\,.$$

Form the evident commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\;h\;} & X \times (H\backslash G) \\ & f\searrow \quad \swarrow\pi & \\ & X & \end{array}$$

where $\pi\colon X \times (H\backslash G) \to X$ is the projection.

Define an isomorphism

$$K^i(X, G) \to K^i_G(T^*[X \times H\backslash G] \oplus \pi^* T^* X) \tag{1}$$

by

$$(Z, \xi, f) \to h_!(\xi)\,.$$

Now $T^*[X \times H\backslash G] \oplus \pi^*T^*X = \pi^*T^*X \oplus \pi^*T^*X \oplus \rho^*T^*(H\backslash G)$, where $\rho\colon X \times H\backslash G \to H\backslash G$ is the projection. $\pi^*T^*X \oplus \pi^*T^*X$ has a G-invariant Spin^c-structure. Hence the Thom isomorphism theorem applies to give an isomorphism

$$(2) \qquad K_G^i(T^*[X \times H\backslash G] \oplus \pi^*T^*X) \to K_G^i(\rho^*T^*(H\backslash G)).$$

Next, there is the identification

$$[X \times (\mathfrak{h}\backslash\mathfrak{g})^*] \times_H G = \rho^*T^*(H\backslash G).$$

This identification gives an induction isomorphism

$$(3) \qquad K_H^i[X \times (\mathfrak{h}\backslash\mathfrak{g})^*] \to K_G^i(\rho^*T^*(H\backslash G)).$$

Starting with an H-vector bundle E on $X \times (\mathfrak{h}\backslash\mathfrak{g})^*$ the induction isomorphism takes E to $E \times_H G$. Combining the isomorphisms (1), (2), (3) proves the proposition. $\square$

REMARK 7. Of special interest is the case when X is a point. By the above proposition

$$K^\epsilon(\cdot, G) \cong R(\widetilde{H})_-$$
$$K^{1+\epsilon}(\cdot, G) = 0.$$

Here $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$ and $R(\widetilde{H})_- = K^0_{\widetilde{H}}(\cdot)_-$ is the free abelian group with one generator for each irreducible representation of $\widetilde{H}$ which is *not* a representation of H. If the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin, then there is an identification $R(\widetilde{H})_- = R(H)$. The second-named author (A. Connes) and independently G. G. Kasparov [20] have conjectured that Dirac induction gives an isomorphism

$$K_\epsilon[C^*G] \cong R(\widetilde{H})_-$$
$$K_{1+\epsilon}[C^*G] = 0.$$

For connected complex semi-simple groups M. Pennington and R. Plymen [25], [28], have verified this conjecture. These results of M. Pennington and R. Plymen combined with the proposition of this section verify the isomorphism conjecture stated in §2 above in a number of interesting cases. Note that (due to the proposition of this section) the Connes-Kasparov conjecture on K_*C^*G is a special case of the isomorphism conjecture of §2.

Let G be a connected semi-simple Lie group with finite center. The lemma of this section elucidates the role of $H\backslash G$ in the Atiyah-Schmid geometric construction of the discrete series [4]. Atiyah and Schmid obtain the discrete series representations by using the Dirac operator on $H\backslash G$. As noted in the introduction $K_0[C^*G]$ contains a free abelian group with one generator for each (irreducible) discrete series representation. By the lemma, however, all of $K^*(\cdot, G)$ is obtained from $H\backslash G$. If (as conjectured in §2 above) $K^*(\cdot, G) \cong K_*(C^*G)$, then not only the discrete series, but *all* of $K_*(C^*G)$ can be obtained from $H\backslash G$.

At this juncture one might ask, "*Why not simply define* $K^i(X, G) = K^i_H(X)$ *?*" We believe that there are compelling reasons for not doing this. First, this misses the dimension-shift by $\epsilon = \dim(H\backslash G)$. Second, this overlooks the issue of whether or not the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c. Third, this greatly obscures the relation of K-theory to index theory. Finally, in the case of discrete groups and foliations there is no maximal compact subgroup so that if this were done there would be no unified theory for Lie groups, discrete groups, and foliations.

6. Discrete groups: Chern character

In this section G is a discrete group which is either finite or countable infinite. For a G-manifold X, $K^*(X, G)$ was defined in §2 above. As in §3 there is the natural map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G),$$

where $\tau = [EG \times T^*X]/G$.

PROPOSITION 1. *Let G be a discrete group and X a G-manifold. Then*

$$K_*^\tau([EG \times X]/G) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(X, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.

REMARK 2. When X is a point, Proposition 1 asserts that

$$K_*(BG) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(\cdot, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.