2.1 Dirichlet characters

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

Bernoulli polynomials in τ for similar values of the variable s. These functions are designed so that $L_p(s,0;\chi) = L_p(s;\chi)$. The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those $\tau \in \overline{\mathbf{Q}}_p$ with $|\tau|_p \leq 1$. To complete the derivation we show that there exist functions $L_p(s,\tau;\chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, such that for every sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to some $\tau \in \mathbf{C}_p$, the sequence $\{L_p(1-n,\tau_i;\chi)\}_{i=0}^{\infty}$, with $n \in \mathbf{Z}$, $n \geq 1$, converges to $L_p(1-n,\tau;\chi)$. Thus for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, the function $L_p(s,\tau;\chi)$ must interpolate the appropriate expressions involving generalized Bernoulli polynomials for $s = 1-n$, $n \in \mathbf{Z}$, $n \geq 1$.

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

2.1 Dirichlet characters

For $n \in \mathbf{Z}$, $n \geq 1$, a Dirichlet character to the modulus n is a multiplicative map $\chi : \mathbf{Z} \to \mathbf{C}$ such that $\chi(a+n) = \chi(a)$ for all $a \in \mathbf{Z}$, and $\chi(a) = 0$ if and only if $(a,n) \neq 1$. Since $a^{\phi(n)} \equiv 1 \pmod{n}$ for all a such that $(a,n) = 1$, $\chi(a)$ must be a root of unity for such a.

If χ is a Dirichlet character to the modulus n, then for any positive multiple m of n we can induce a Dirichlet character ψ to the modulus m according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a,m) = 1 \\ 0, & \text{if } (a,m) \neq 1. \end{cases}$$

The minimum modulus n for which a character χ cannot be induced from some character to the modulus m, $m < n$, is called the conductor of χ, denoted f_χ. We shall assume that each χ is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters χ and ψ having conductors f_χ and f_ψ, respectively, we define the product, $\chi\psi$, to be the primitive character with $\chi\psi(a) = \chi(a)\psi(a)$ for all $a \in \mathbf{Z}$ such that $(a, f_\chi f_\psi) = 1$. Note that there may exist some values of a such that $\chi\psi(a) \neq \chi(a)\psi(a)$, due to the fact that our definition requires $\chi\psi$ to be a primitive character. The conductor $f_{\chi\psi}$ then divides $\operatorname{lcm}(f_\chi, f_\psi)$. With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character $\chi = 1$, having conductor $f_1 = 1$. The inverse of the character χ is the character $\chi^{-1} = \overline{\chi}$, the map of complex conjugates of the values of χ.

Since any Dirichlet character χ is multiplicative, we must have $\chi(-1) = \pm 1$. A character χ is said to be odd if $\chi(-1) = -1$, and even if $\chi(-1) = 1$.

2.2 Generalized Bernoulli polynomials

Let χ be a Dirichlet character with conductor f_χ. Then we define the functions, $B_{n,\chi}(t)$, $n \in \mathbf{Z}$, $n \geq 0$, by the generating function

(1) $$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t)\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with χ, $B_{n,\chi}$, $n \in \mathbf{Z}$, $n \geq 0$, by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that $B_{n,\chi}(0) = B_{n,\chi}$. Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t)\frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!},$$

and from this we obtain

(2) $$B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions $B_{n,\chi}(t)$, defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with χ. Let $\mathbf{Z}[\chi]$ denote the ring generated over $\mathbf{Z}$ by all the values $\chi(a)$, $a \in \mathbf{Z}$, and $\mathbf{Q}(\chi)$ the field generated over $\mathbf{Q}$ by all such values. Then it can be shown that $f_\chi B_{n,\chi}$ must be in $\mathbf{Z}[\chi]$ for each $n \geq 0$ whenever $\chi \neq 1$. In general, we have $B_{n,\chi} \in \mathbf{Q}(\chi)$ for each $n \geq 0$, and so $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$. The polynomials $B_{n,\chi}(t)$ exhibit the property that, for all $n \geq 0$,

(3) $$B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever $\chi \neq 1$. Thus $B_{n,\chi}(t)$, for $\chi \neq 1$, is either an even function or an odd function according to whether $(-1)^n\chi(-1)$ is 1 or -1. From (3) we obtain