

## 2.1 Dirichlet characters

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

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Bernoulli polynomials in  $\tau$  for similar values of the variable  $s$ . These functions are designed so that  $L_p(s, 0; \chi) = L_p(s; \chi)$ . The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those  $\tau \in \overline{\mathbf{Q}}_p$  with  $|\tau|_p \leq 1$ . To complete the derivation we show that there exist functions  $L_p(s, \tau; \chi)$  for all  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , such that for every sequence  $\{\tau_i\}_{i=0}^\infty$  in  $\overline{\mathbf{Q}}_p$ , with  $|\tau_i|_p \leq 1$ , converging to some  $\tau \in \mathbf{C}_p$ , the sequence  $\{L_p(1-n, \tau_i; \chi)\}_{i=0}^\infty$ , with  $n \in \mathbf{Z}$ ,  $n \geq 1$ , converges to  $L_p(1-n, \tau; \chi)$ . Thus for each  $\tau \in \mathbf{C}_p$ ,  $|\tau|_p \leq 1$ , the function  $L_p(s, \tau; \chi)$  must interpolate the appropriate expressions involving generalized Bernoulli polynomials for  $s = 1 - n$ ,  $n \in \mathbf{Z}$ ,  $n \geq 1$ .

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

## 2.1 DIRICHLET CHARACTERS

For  $n \in \mathbf{Z}$ ,  $n \geq 1$ , a Dirichlet character to the modulus  $n$  is a multiplicative map  $\chi : \mathbf{Z} \rightarrow \mathbf{C}$  such that  $\chi(a+n) = \chi(a)$  for all  $a \in \mathbf{Z}$ , and  $\chi(a) = 0$  if and only if  $(a, n) \neq 1$ . Since  $a^{\phi(n)} \equiv 1 \pmod{n}$  for all  $a$  such that  $(a, n) = 1$ ,  $\chi(a)$  must be a root of unity for such  $a$ .

If  $\chi$  is a Dirichlet character to the modulus  $n$ , then for any positive multiple  $m$  of  $n$  we can induce a Dirichlet character  $\psi$  to the modulus  $m$  according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a, m) = 1 \\ 0, & \text{if } (a, m) \neq 1. \end{cases}$$

The minimum modulus  $n$  for which a character  $\chi$  cannot be induced from some character to the modulus  $m$ ,  $m < n$ , is called the conductor of  $\chi$ , denoted  $f_\chi$ . We shall assume that each  $\chi$  is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters  $\chi$  and  $\psi$  having conductors  $f_\chi$  and  $f_\psi$ , respectively, we define the product,  $\chi\psi$ , to be the primitive character with  $\chi\psi(a) = \chi(a)\psi(a)$  for all  $a \in \mathbf{Z}$  such that  $(a, f_\chi f_\psi) = 1$ . Note that there may exist some values of  $a$  such that  $\chi\psi(a) \neq \chi(a)\psi(a)$ , due to the fact that our definition requires  $\chi\psi$  to be a primitive character. The conductor  $f_{\chi\psi}$  then divides  $\text{lcm}(f_\chi, f_\psi)$ . With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character  $\chi = 1$ , having conductor  $f_1 = 1$ . The inverse of the character  $\chi$  is the character  $\chi^{-1} = \overline{\chi}$ , the map of complex conjugates of the values of  $\chi$ .

Since any Dirichlet character  $\chi$  is multiplicative, we must have  $\chi(-1) = \pm 1$ . A character  $\chi$  is said to be odd if  $\chi(-1) = -1$ , and even if  $\chi(-1) = 1$ .

### 2.2 GENERALIZED BERNOULLI POLYNOMIALS

Let  $\chi$  be a Dirichlet character with conductor  $f_\chi$ . Then we define the functions,  $B_{n,\chi}(t)$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by the generating function

$$(1) \quad \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with  $\chi$ ,  $B_{n,\chi}$ ,  $n \in \mathbf{Z}$ ,  $n \geq 0$ , by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that  $B_{n,\chi}(0) = B_{n,\chi}$ . Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!},$$

and from this we obtain

$$(2) \quad B_{n,\chi}(t) = \sum_{m=0}^n \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions  $B_{n,\chi}(t)$ , defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with  $\chi$ . Let  $\mathbf{Z}[\chi]$  denote the ring generated over  $\mathbf{Z}$  by all the values  $\chi(a)$ ,  $a \in \mathbf{Z}$ , and  $\mathbf{Q}(\chi)$  the field generated over  $\mathbf{Q}$  by all such values. Then it can be shown that  $f_\chi B_{n,\chi}$  must be in  $\mathbf{Z}[\chi]$  for each  $n \geq 0$  whenever  $\chi \neq 1$ . In general, we have  $B_{n,\chi} \in \mathbf{Q}(\chi)$  for each  $n \geq 0$ , and so  $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$ . The polynomials  $B_{n,\chi}(t)$  exhibit the property that, for all  $n \geq 0$ ,

$$(3) \quad B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever  $\chi \neq 1$ . Thus  $B_{n,\chi}(t)$ , for  $\chi \neq 1$ , is either an even function or an odd function according to whether  $(-1)^n \chi(-1)$  is 1 or  $-1$ . From (3) we obtain