2.2 Generalized Bernoulli polynomials

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Since any Dirichlet character χ is multiplicative, we must have $\chi(-1)=\pm 1$. A character χ is said to be odd if $\chi(-1)=-1$, and even if $\chi(-1)=1$.

2.2 Generalized Bernoulli polynomials

Let χ be a Dirichlet character with conductor f_χ. Then we define the functions, $B_{n,\chi}(t)$, $n\in\mathbf{Z}$, $n\geq 0$, by the generating function

$$\sum_{a=1}^{f_\chi}\frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x}-1}=\sum_{n=0}^{\infty}B_{n,\chi}(t)\frac{x^n}{n!},\qquad |x|<\frac{2\pi}{f_\chi}. \tag{1}$$

We define the generalized Bernoulli numbers associated with χ, $B_{n,\chi}$, $n\in\mathbf{Z}$, $n\geq 0$, by

$$\sum_{a=1}^{f_\chi}\frac{\chi(a)xe^{ax}}{e^{f_\chi x}-1}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{x^n}{n!},\qquad |x|<\frac{2\pi}{f_\chi},$$

so that $B_{n,\chi}(0)=B_{n,\chi}$. Note that

$$\sum_{a=1}^{f_\chi}\frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x}-1}=e^{tx}\sum_{a=1}^{f_\chi}\frac{\chi(a)xe^{ax}}{e^{f_\chi x}-1},$$

which implies that

$$\sum_{n=0}^{\infty}B_{n,\chi}(t)\frac{x^n}{n!}=e^{tx}\sum_{n=0}^{\infty}B_{n,\chi}\frac{x^n}{n!},$$

and from this we obtain

$$B_{n,\chi}(t)=\sum_{m=0}^{n}\binom{n}{m}B_{n-m,\chi}t^m. \tag{2}$$

Thus the functions $B_{n,\chi}(t)$, defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with χ. Let $\mathbf{Z}[\chi]$ denote the ring generated over $\mathbf{Z}$ by all the values $\chi(a)$, $a\in\mathbf{Z}$, and $\mathbf{Q}(\chi)$ the field generated over $\mathbf{Q}$ by all such values. Then it can be shown that $f_\chi B_{n,\chi}$ must be in $\mathbf{Z}[\chi]$ for each $n\geq 0$ whenever $\chi\neq 1$. In general, we have $B_{n,\chi}\in\mathbf{Q}(\chi)$ for each $n\geq 0$, and so $B_{n,\chi}(t)\in\mathbf{Q}(\chi)[t]$. The polynomials $B_{n,\chi}(t)$ exhibit the property that, for all $n\geq 0$,

$$B_{n,\chi}(-t)=(-1)^n\chi(-1)B_{n,\chi}(t), \tag{3}$$

whenever $\chi\neq 1$. Thus $B_{n,\chi}(t)$, for $\chi\neq 1$, is either an even function or an odd function according to whether $(-1)^n\chi(-1)$ is 1 or -1. From (3) we obtain

$$B_{n,\chi} = (-1)^n \chi(-1) B_{n,\chi}\,,$$

and so $B_{n,\chi} = 0$ whenever n is even and χ is odd, or whenever n is odd and χ is even, $\chi \neq 1$. Another property that the polynomials satisfy is that for $m \in \mathbf{Z}$, $m \geq 1$,

$$B_{n,\chi}(mf_\chi + t) - B_{n,\chi}(t) = n \sum_{a=1}^{mf_\chi} \chi(a)(a+t)^{n-1}, \tag{4}$$

for all $n \geq 0$. This can be derived from (1). Note that for $\chi = 1$ and $t = 0$ this becomes

$$\frac{1}{n}\left(B_{n,1}(m) - B_{n,1}\right) = \sum_{a=1}^{m} a^{n-1}.$$

If $\chi \neq 1$, then it can be shown that $\sum_{a=1}^{f_\chi} \chi(a) = 0$, and from the above relations we can derive

$$B_{0,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a)$$

for all χ. Therefore

$$B_{0,\chi} = \begin{cases} 0, & \text{if } \chi \neq 1 \\ 1, & \text{if } \chi = 1\,. \end{cases}$$

The ordinary Bernoulli polynomials, $B_n(t)$, $n \in \mathbf{Z}$, $n \geq 0$, are defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \qquad |x| < 2\pi, \tag{5}$$

and the Bernoulli numbers, B_n, $n \in \mathbf{Z}$, $n \geq 0$,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi\,.$$

From this we obtain the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots,$ with $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m\,.$$

Note that we again have the relations $B_n(0) = B_n$ and

$$B_n(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} t^m,$$

as we did for the generalized Bernoulli polynomials.

Some of the more important properties of Bernoulli polynomials are that

$$B_n(t+1) - B_n(t) = nt^{n-1} \tag{6}$$

for all $n \geq 1$, and

$$B_n(1-t) = (-1)^n B_n(t)$$

for $n \geq 0$. Each of these results can be derived from the generating function (5) above.

Similar to (4) for the generalized Bernoulli polynomials, whenever $m, n \in \mathbf{Z}$, $m \geq 1$, $n \geq 1$,

$$\frac{1}{n}\left(B_n(m) - B_n\right) = \sum_{a=0}^{m-1} a^{n-1},$$

where we take 0^0 to be 1 in the case of $a = 0$ and $n = 1$. Note that this can be derived from (6) since

$$B_n(m) - B_n = \sum_{a=0}^{m-1} \left(B_n(a+1) - B_n(a)\right).$$

The Bernoulli numbers are rational numbers, and, in fact, the von Staudt-Clausen theorem states that for even $n \geq 2$,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus the denominator of each B_n must be square-free.

The ordinary Bernoulli numbers are related to the generalized Bernoulli numbers in that for $\chi = 1$ we have

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n,1} \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and since

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1},$$

we see that $B_{n,1} = B_n$ for all $n \neq 1$, and $B_{1,1} = -B_1$. In fact, this can be written as $B_{n,1} = (-1)^n B_n$, and for the polynomials, $B_{n,1}(t) = (-1)^n B_n(-t)$.