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2.3 DIRICHLET L-FUNCTIONS

For χ a Dirichlet character with conductor f_χ, the Dirichlet L-function for χ is defined by

$$L(s;\chi)=\sum_{b=1}^{\infty}\frac{\chi(b)}{b^s},$$

for $s\in\mathbf{C}$ such that $\Re(s)>1$. Note that $L(s;\chi)$ can be continued analytically to all of $\mathbf{C}$, except for a pole of order 1 at $s=1$ when $\chi=1$.

Let $\tau(\chi)$ be a Gauss sum,

$$\tau(\chi)=\sum_{a=1}^{f_\chi}\chi(a)e^{2\pi ia/f_\chi},$$

where $i^2=-1$, and let

$$\delta_\chi=\begin{cases}0, & \text{if } \chi(-1)=1\\ 1, & \text{if } \chi(-1)=-1\,.\end{cases}$$

Then $L(s;\chi)$ satisfies the functional equation

$$(7)\quad \left(\frac{f_\chi}{\pi}\right)^{s/2}\Gamma\left(\frac{s+\delta_\chi}{2}\right)L(s;\chi)=W_\chi\left(\frac{f_\chi}{\pi}\right)^{(1-s)/2}\Gamma\left(\frac{1-s+\delta_\chi}{2}\right)L(1-s;\overline{\chi}),$$

where $\Gamma(s)$ is the gamma function, and $W_\chi=\frac{\tau(\chi)}{i^{\delta_\chi}\cdot\sqrt{f_\chi}}$, having the property that $|W_\chi|=1$. Since $\Gamma(s)$ has simple poles at the negative integers, $L(s;\chi)$ must be zero for $s=1-n$, where $n\in\mathbf{Z}$, $n\geq 1$, such that $n\not\equiv\delta_\chi \pmod 2$, except when $\chi=1$ and $n=1$. $L(s;\chi)$ can also be described by means of the Euler product $L(s;\chi)=\prod_{p\text{ prime}}\left(1-\chi(p)p^{-s}\right)^{-1}$, for $s\in\mathbf{C}$ such that $\Re(s)>1$. Thus $L(s;\chi)\neq 0$ in this domain.

The generalized Bernoulli numbers, $B_{n,\chi}$, and the Dirichlet L-function, $L(s;\chi)$, share the following relationship, a proof of this being found in [13]:

THEOREM 2.1. *Let χ be a Dirichlet character, and let $n\in\mathbf{Z}$, $n\geq 1$. Then $L(1-n;\chi)=-\frac{1}{n}B_{n,\chi}$.*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.