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The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where $\mathbf{L}(M)$ is the total $\mathbf{L}$ class of TM and a is any element in $H^*(BG; \mathbf{Q})$.

Kasparov [19] and Miscenko-Fomenko [21] [22] define a map

$$K_0(BG) \to K_0\, C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0\, C^*G \\ & \searrow \quad \swarrow & \\ & K^0(\cdot, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of K-homology given in [9] is being used.) □

COROLLARY 5. *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg* [30].

For the same reason our conjecture implies the stable[1]) form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the C^*-algebra of the projective regular representation of a discrete group.

All of § 2 adapts to the projective situation where together with the G-manifold X one is given a 2-cocycle $\gamma \in Z^2(X \rtimes G, S^1)$. For simplicity we

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[1]) Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture Also, Max Karoubi and I have proved that the usual (i e complex K-theory) form of Baum-Connes implies the real form of Baum-Connes

shall stick to the case $X = \mathrm{pt} = \cdot$ and G discrete $= \Gamma$; then $\gamma \in Z^2(\Gamma, S^1)$ is a map: $\Gamma \times \Gamma \to S^1$ such that:

$$\gamma(g_2, g_3)\,\gamma(g_1 g_2, g_3)^{-1}\,\gamma(g_1, g_2 g_3)\,\gamma(g_1, g_2)^{-1} = 1 \quad \text{for every } g_1, g_2, g_3 \in \Gamma\,.$$

Given a *proper* Γ-manifold Z, a (Γ, γ)-vector bundle on Z is a smooth (complex) vector bundle E on Z together with a smooth map $E \times \Gamma \to E$ such that (with $\pi\colon E \to Z$ the projection):

a) $\pi(\xi g) = \pi(\xi) g$ for each $\xi \in E$, $g \in \Gamma$;

b) $\xi(g_1 g_2) = \gamma(g_1, g_2)\,(\xi g_1) g_2$ for each $g_1, g_2 \in \Gamma$.

In b), $\gamma(g_1, g_2) \in S^1$ is viewed as a complex number of modulus 1. As in §2, we let $V^0_{(\Gamma,\gamma)}(Z)$ be the collection of triples (E_0, E_1, σ) where E_0, E_1 are (Γ, γ)-vector bundles over Z and σ is a smooth morphism of vector bundles such that:

1) $\sigma(\xi g) = \sigma(\xi) g$ for each $\xi \in E_0$, $g \in \Gamma$;

2) Support(σ) is Γ-compact.

The groups $K^i_{(\Gamma,\gamma)}(Z)$ are then defined as in [5], [31]. The Thom isomorphism as formulated in §2 still holds in this context, and this allows us to define Gysin maps:

$$h!\colon K^i_{(\Gamma,\gamma)}(T^*Z_1) \to K^i_{(\Gamma,\gamma)}(T^*Z_2)$$

for a Γ-map h of the proper Γ-manifold Z_1 to the proper Γ-manifold Z_2.

Thus as in §2 we can define the geometric group also in this twisted situation, we denote it by $K^*_\gamma(X, G)$ in general, and $= K^*_\gamma(\cdot, \Gamma)$ in our special case.

Let then $C^*_r(\Gamma, \gamma)$ be the reduced C^*-algebra of the pair (Γ, γ), i.e. the C^*-algebra generated in $\ell^2(\Gamma)$ by the projective regular representation λ of Γ:

$$(\lambda(g)\xi)(g') = \gamma(g, g^{-1}g')\,\xi(g^{-1}g')\,.$$

As in §2 we get a map μ from $K^*_\gamma(\mathrm{pt}, \Gamma)$ to $K_*(C^*_r(\Gamma, \gamma))$, where $\mu(Z, \xi)$ is the analytical index of the K-cocycle $(Z, \xi) \in V^*_{(\Gamma,\gamma)}(T^*Z)$. The only part of the construction which is modified by the presence of γ is that of the C^*-module over $C^*_r(\Gamma, \gamma)$ attached to a (Γ, γ)-bundle E on the proper Γ-manifold Z. More precisely, one starts with the space $C_c(Z, E \otimes \Omega^{1/2})$ of compactly supported continuous $\frac{1}{2}$-density sections of E and, after choosing a Γ-invariant metric on E, one defines:

$$\langle \xi, \eta \rangle (g) = \int_X \langle \xi_x\,, (\eta_{xg})\, g^{-1} \rangle \quad \text{for each } g \in \Gamma\,,$$

which gives a $C_c(\Gamma)$-valued sesquilinear form on $C_c(Z,E\otimes\Omega^{1/2})$. One checks that for any $\xi\in C_c(Z,E\otimes\Omega^{1/2})$, $\langle\xi,\xi\rangle$ is a *positive* element of $C_r^*(\Gamma)$, since for any $\eta\in\ell^2(\Gamma)$ one has:

$$\begin{aligned}\langle\eta,\lambda(\langle\xi,\xi\rangle)\eta\rangle &= \sum\overline{\eta}(g)\,\langle\xi,\xi\rangle\,(h)\,(\lambda(h)\eta)\,(g)\\ &= \sum\gamma(h,h^{-1}g)\overline{\eta}(g)\,\eta(h^{-1}g)\int_X\langle\xi_x,(\xi_x h)\,h^{-1}\rangle\\ &= \sum\overline{\eta}(g)\,\eta(h^{-1}g)\int_X\langle(\xi_{xg^{-1}})\,g,(\xi_{xg^{-1}h})\,h^{-1}g\rangle\geq 0\,.\end{aligned}$$

Then, by completion with respect to the norm $\|\langle\xi,\xi\rangle\|^{1/2}$, one gets a C^*-module over $C_r^*(\Gamma,\gamma)$, which we denote by $L^2(Z,E)$. The right action is given by:

$$(\xi f)\,(x)=\sum_\Gamma(\xi_{xg^{-1}})gf(g)\quad\text{for each }\xi\in C_c(Z,E\otimes\Omega^{1/2}),\ f\in C_c(\Gamma)\,.$$

Next, we can choose a Γ-invariant Riemannian metric on Z, represent every class in $K^0_{(\Gamma,\gamma)}(T^*Z)$ by a pair E_0,E_1 of (Γ,γ)-hermitian bundles on Z and a symbol σ which is an isomorphism of the pull back of E_0 to S^*Z to that of E_1, and is independent of ξ, $\pi(\xi)=z$, outside a Γ-compact subset of Z. Letting P_σ be the corresponding order 0 pseudo-differential operator, one gets a Kasparov $\big(\mathbf{C},C_r^*(\Gamma,\gamma)\big)$-bimodule: the triple $(L^2(Z,E_0),\ L^2(Z,E_1),P_\sigma)$ which gives an element of $K_0(C_r^*(\Gamma,\gamma))$. It is important to give another description of the map $\mu\colon K^0_{(\Gamma,\gamma)}(T^*Z)\to K_0(C_r^*(\Gamma,\gamma))$, using Kasparov products.

PROPOSITION 1. a) *Let X be a proper Γ-manifold, then $K^i_{(\Gamma,\gamma)}(X)$ is canonically isomorphic to $K_i(C_0(X)\rtimes_\gamma\Gamma)$, where $C_0(X)\rtimes_\gamma\Gamma$ is the twisted crossed product of $C_0(X)$ by Γ.*

b) (Compare [19]). *For any C^*-algebras A,B on which Γ acts by automorphisms, one has a natural map from $KK_\Gamma(A,B)$ to $KK(A\rtimes_\gamma\Gamma,B\rtimes_\gamma\Gamma)$.*

Proof. a) One can consider $A=C_0(X)\rtimes_\gamma\Gamma$ as the C^*-algebra of the groupoid $X\rtimes\Gamma=G$ with units $G^{(0)}=X$, source and range maps $s(x,g)=xg$, $r(x,g)=x$ and composition $(x,g)\cdot(x',g')=(x,gg')$ with the 2-cocycle $\gamma\circ\pi$ where π is the natural homomorphism $G\to\Gamma$: $\pi(x,g)=g$.

Thus A is the completion of this convolution algebra $C_c(G)$:

$$(f_1*f_2)\,(x,g)=\sum_\Gamma f_1(x,h)f_2(xh,h^{-1}g)\,\gamma(h,h^{-1}g)$$

$$f^*(x,g)=\overline{f}(xg,g^{-1})$$

with the norm $\|f\| = \operatorname{Sup} \|\pi_x(f)\|$, where for each $x \in X$ the representation π_x of $C_c(G)$ in $\ell^2(\Gamma)$ is given by:

$$(\pi_x(f)\,\xi)\,(g) = \sum_{\Gamma} f(xg^{-1}, h)\,\xi(h^{-1}\,g)\,\gamma(h, h^{-1}\,g) \quad \text{for each } \xi \in \ell^2(\Gamma)\,.$$

Now, given a (Γ, γ)-vector bundle E on X, one can endow E with a Γ-invariant hermitian metric and define a C^*-module $\mathcal{E}$ over $A = C_0(X) \rtimes_\gamma \Gamma$ as follows. For any $\xi, \eta \in C_c(X, E)$ let $\langle \xi, \eta \rangle \in C_c(X \rtimes \Gamma)$ be given by $\langle \xi, \eta \rangle(x, g) = \langle \xi_x\, g, \eta_{xg} \rangle$; then $\langle \xi, \xi \rangle$ is a positive element of $A = C_0(X) \rtimes_\gamma \Gamma$, since for any $\eta \in \ell^2(\Gamma)$ and $x \in X$ one has:

$$\langle \eta, \pi_x(\langle \xi, \xi \rangle)\,\eta \rangle =$$
$$\sum \sum \langle \xi_{xg^{-1}}\, h, \xi_{xg^{-1}}\, h \rangle\, \eta(h^{-1}\,g)\,\overline{\eta}(g)\,\gamma(h, h^{-1}\,g) = \langle \alpha, \alpha \rangle \geq 0\,,$$

where $\alpha = \sum (\xi_{xg^{-1}})g\,\eta(g) \in E_x$.

Let $\mathcal{E}$ be the completion of $C_c(X, E)$ with the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|$; then $\mathcal{E}$ is a C^*-module over A, with:

$$(\xi f)\,(x) = \sum f(xg^{-1}, g)\,\xi(xg')g \quad \text{for every } f \in C_c(X \rtimes \Gamma),\ \xi \in C_0(X, E)\,.$$

(One easily checks that $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle * f$ and that this right action of $C_c(X \rtimes \Gamma)$ extends to an action of A.)

The equality $(\eta \langle \eta, \xi \rangle)\,(x) = \sum \langle (\eta_{xg^{-1}})\,g, \xi_x \rangle (\eta_{xg^{-1}})\,g$ shows that any endomorphism σ of the vector bundle E which commutes with Γ and has Γ-compact support defines an A-compact endomorphism of $\mathcal{E}$ by the equality: $(T\xi)\,(x) = \sigma(x)\,\xi(x)$ for every $x \in X$. Thus, to any triple $(E_0, E_1, \sigma) \in V^0_{(\Gamma,\gamma)}(X)$ corresponds an element of $KK(\mathbf{C}, A)$, $A = C_0(X) \rtimes_\gamma \Gamma$, which obviously depends only upon the class of the triple in $K^0_{(\Gamma,\gamma)}(X)$. Let us prove that this map is an isomorphism assuming that Γ *is torsion free*. We may then assume that X is Γ-compact. We claim first that $A = C_0(X) \rtimes_\gamma \Gamma$ is Morita equivalent to a C^*-algebra with unit. Indeed, with $V = X/\Gamma$, A is the C^*-algebra of the continuous field of elementary C^*-algebras $A_t = C_0(\pi^{-1}(t)) \rtimes_\gamma \Gamma$, where $\pi\colon X \to X/\Gamma = V$ is the projection. By a simple computation, one gets that the Dixmier-Douady obstruction $\delta(A) \in H^3(V, \mathbf{Z})$ is given by $\delta(A) = \phi^*(\partial\gamma)$ where $\phi\colon V \to B\Gamma$ is the classifying map, and $\partial\gamma \in H^3(B\Gamma, \mathbf{Z})$ is the boundary of $\gamma \in H^2(B\Gamma, S^1) = H^2(\Gamma, S^1)$ in the exact sequence:

$$H^2(\Gamma, \mathbf{Z}) \to H^2(\Gamma, \mathbf{R}) \to H^2(\Gamma, S^1) \xrightarrow{\partial} H^3(\Gamma, \mathbf{Z}) \to H^3(\Gamma, \mathbf{R}) \to \ldots$$

In particular $\delta(A)$ is a torsion element in $H^3(V, \mathbf{Z})$ so that there exists a bundle of matrix algebras over V with the same Dixmier-Douady obstruction and A is Morita equivalent to a unital C^*-algebra. It follows then that $K_0(A)$

is obtained from C^*-modules $\mathcal{E}$ over A with the property $\mathrm{id}_{\mathcal{E}} \in \mathrm{End}^0_A(\mathcal{E})$, i.e. all endomorphisms of $\mathcal{E}$ are A-compact. Finally, the above construction sets up a surjective map from (Γ,γ)-vector bundles on X to C^*-modules over A with the above property. Given $\mathcal{E}$, the fiber E_x of the corresponding vector bundle is:

$$E_x = \mathcal{E} \,\widehat{\otimes}_A\, \ell^2(\Gamma)$$

where $A = C_0(X) \rtimes_\gamma \Gamma$ acts in $\ell^2(\Gamma)$ by the representation π_x. Since $\pi_x(A) \subset$ Compacts, one gets that E_x is a finite dimensional Hilbert space.

b) The proof is the same as in [19], one defines for any Γ-equivariant C^*-module $\mathcal{E}$ over B the crossed product $\mathcal{E} \rtimes_\gamma \Gamma$ twisted by the 2-cocycle γ. □

We can now state:

THEOREM 2. *For any element x of $K^0_{(\Gamma,\gamma)}(T^*Z) = K_0(A)$ (where $A = C_0(T^*Z) \rtimes_\gamma \Gamma$, and Z a proper Γ-manifold), one has:*

$$\mu(x) = x \otimes j_{(\Gamma,\gamma)}(D),$$

where $D \in KK_\Gamma(C_0(T^*Z), \mathbf{C})$ is the class of the Dirac operator.

Note that $x \in KK(\mathbf{C}, C_0(T^*Z) \rtimes_\gamma \Gamma)$ and that

$$j_{(\Gamma,\gamma)}(D) \in KK\big(C_0(T^*Z)) \rtimes_\gamma \Gamma, C^*_r(\Gamma,\gamma)\big),$$

so that the above equality is meaningful. The proof is straightforward.

To show how to use this theorem, we shall combine it with the recent result of G. G. Kasparov ([19]) to compute $K_i(C^*_r(\Gamma,\gamma))$ in the following example: we let $\Gamma = \pi_1(M)$ be the fundamental group of a Riemann surface M with genus > 1. From the exact sequence $0 \to H^2(\Gamma,\mathbf{Z}) \to H^2(\Gamma,\mathbf{R}) \to H^2(\Gamma,S^1) \to 0$ one gets $H^2(\Gamma,S^1) = \mathbf{R}/\mathbf{Z}$, so that there are many non trivial cocycles in this example. The geometric group $K^i_\gamma(\mathrm{pt},\Gamma)$ is easily determined: since the universal cover $\widetilde{M}$ of M (the Poincaré disc) is a final object in the category of proper Γ-manifolds, and homotopy classes of Γ-maps, it is enough to compute $K^i_{(\Gamma,\gamma)}(T^*\widetilde{M})$. Since $\widetilde{M}$ has a Γ-invariant Spin^c-structure, the Thom isomorphism hence gives: $K^i_\gamma(\mathrm{pt},\Gamma) = K^i_{(\Gamma,\gamma)}(\widetilde{M})$. By Proposition 1, one has $K^i_{(\Gamma,\gamma)}(\widetilde{M}) = K_i(C_0(\widetilde{M}) \rtimes_\gamma \Gamma)$ and the latter C^*-algebra is Morita equivalent to $C(M)$ (see the proof of a) in Proposition 1). Thus we get: $K^0_\gamma(\mathrm{pt},\Gamma) = \mathbf{Z}^2$, $K^1_\gamma(\mathrm{pt},\Gamma) = \mathbf{Z}^{2g}$.

THEOREM 3. *Let Γ be the fundamental group of a Riemann surface of genus >1, and $\gamma \in H^2(\Gamma, S^1)$, then the map $\mu\colon K^*_\gamma(\mathrm{pt}, \Gamma) \to K_*(C^*_r(\Gamma, \gamma))$ is an isomorphism.*

Proof. Let $D \in KK_G(C_0(U), \mathbf{C})$ be the $G = PSL(2, \mathbf{R})$ equivariant Dirac operator on the Poincaré disc $U = G/G_c$ (cf. [19]). Identify $\widetilde{M}$ with U and Γ with a subgroup of G. Then by Proposition 1 b) and Theorem 2 it is enough to show that the restriction of D to an element of $KK_\Gamma(C_0(U), \mathbf{C})$ is an invertible element. This follows from [19] which shows that D is an invertible element of $KK_G(C_0(U), \mathbf{C})$, and from the multiplicative property of the restriction to subgroups.

We shall now show how to prove that the C^*-algebras $C^*_r(\Gamma, \gamma)$ are pairwise non-isomorphic when γ varies in $H^2(\Gamma, S^1)$. In fact we shall compute in full generality the composition $\zeta \circ \mu$ of the canonical trace ζ on $C^*_r(\Gamma, \gamma)$ (viewed as a map from K_0 to $\mathbf{C}$) with the above map $\mu\colon K^0_\gamma(\mathrm{pt}, \Gamma) \to K_0(C^*_r(\Gamma, \gamma))$.

The computation is a generalization of the index theorem for covering spaces of Atiyah ([3]).

LEMMA 4. *Let Z be a proper Γ-manifold and E a (Γ, γ) vector bundle on Z. There exists a Γ-invariant connection ∇ on E.*

Proof. For any (Γ, γ)-vector bundle F on Z and section $\xi \in C^\infty_c(Z, F)$ let, for $g \in \Gamma$, $g\xi \in C^\infty_c(Z, F)$ be given by: $(g\xi)(x) = (\xi(xg))\, g^{-1} \in F_x$ for every $x \in Z$.

In this way one gets a natural γ-action of Γ on both $C^\infty_c(Z, E)$ and $C^\infty_c(Z, E \otimes T^*Z)$, and one looks for a connection

$$\nabla\colon C^\infty_c(Z, E) \to C^\infty_c(Z, E \otimes T^*Z)$$

such that $\nabla(g\xi) = g(\nabla\xi)$ for every ξ. Let $f \in C^\infty(Z)$, $0 \le f \le 1$, be such that $\sum_\Gamma f(xg) = 1$ for every $x \in Z$ and ∇_0 be a connection on E. Put $\nabla = \sum_\Gamma g^{-1}(f\nabla_0)g$. By construction ∇ is Γ-invariant, moreover each $g^{-1}\nabla_0 g$ is a connection on E thus ∇ is a connection on E. □

Proof of Theorem 3, continued. Assuming now that Z is Γ-compact, let for a Γ-invariant connection ∇ on E, ω_∇ be the canonical differential form on Z which represents locally the Chern character $\mathrm{ch}(E)$. By construction ω_∇ is Γ-invariant and hence determines a cohomology class in Z/Γ. One checks as usual that this class does not depend upon the choice of ∇ and we shall denote it by $[E] \in H^*(Z/\Gamma, \mathbf{R})$. This construction easily extends to give a map ch from $K^0_{(\Gamma,\gamma)}(Z)$ to $H^*(Z/\Gamma, \mathbf{R})$ for any proper Γ-manifold Z. However, in the presence of the 2-cocycle γ the range of this map is *no longer necessarily contained* in $H^*(Z/\Gamma, \mathbf{Q})$.

To be more precise, let us make a few simplifying assumptions and compute exactly the range of this Chern character:

$$\mathrm{ch}\colon K^0_{(\Gamma,\gamma)}(Z) \to H^*(Z/\Gamma, \mathbf{R}).$$

Thus let us assume that Γ is torsion free and that the image of $\gamma \in H^2(\Gamma, S^1)$ in $H^3(\Gamma, \mathbf{Z})$ under the connecting map of the long exact sequence:

$$\ldots \to H^2(\Gamma, \mathbf{Z}) \to H^2(\Gamma, \mathbf{R}) \to H^2(\Gamma, S^1) \to H^3(\Gamma, \mathbf{Z}) \to \ldots$$

is equal to 0 (it is always a torsion element).

Let then $\rho \in H^2(\Gamma, \mathbf{R})$ be such that $e(\rho) = \gamma$ where $e\colon \mathbf{R} \to S^1$ is given by $e(s) = \exp(2\pi i s)$, for each $s \in \mathbf{R}$.

LEMMA 5. a) *Let $\rho \in Z^2(\Gamma, \mathbf{R})$ and Z be a proper Γ-manifold, then there exists a smooth function $c \in C^\infty(Z \rtimes \Gamma)$ such that:*

$$c(x, g_1) + c(xg_1, g_2) = c(x, g_1 g_2) - \rho(g_1, g_2)$$

for every $x \in Z$, $g_1, g_2 \in \Gamma$.

b) *If $\gamma = e(\rho)$ there exists an isomorphism $r\colon K^0_\Gamma(Z) \to K^0_{(\Gamma,\gamma)}(Z)$ making the following diagram commutative:*

$$\begin{array}{ccc} K^0_\Gamma(Z) & \xrightarrow{\ r\ } & K^0_{(\Gamma,\gamma)}(Z) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(Z/\Gamma) & \xrightarrow{\ m\ } & H^*(Z/\Gamma), \end{array}$$

where m is multiplication by the cohomology class $\exp(\phi^*\rho)$ *and where $\phi\colon Z/\Gamma \to B\Gamma$ is the classifying map.*

Proof. a) Let $M = Z/\Gamma$, $\pi\colon Z \to M$ the projection. Since Z is a locally trivial Γ-principal bundle, it is easy to construct c on the open set $\pi^{-1}(U)$ for U small enough. Then one combines such c_U by a smooth partition of unity on M:

$$c(x, g) = \sum \phi_U(\pi(x))\, c_U(x, g).$$

b) Let $c \in C^\infty(Z \rtimes \Gamma)$ be as in a) and let us endow the trivial line bundle on Z (with total space $Z \times \mathbf{C}$) with a structure of (Γ, γ)-bundle. We take:

$$(x, \lambda)g = (xg, e(c(x, g))\lambda).$$

(One has $((x, \lambda)g_1)g_2 = (xg_1g_2, e(c(x, g_1) + c(xg_1, g_2))\lambda) = \gamma^{-1}(g_1, g_2)(x\lambda)(g_1g_2)$.)

Let L be the (Γ, γ)-line bundle on Z thus obtained. It is obvious that tensoring by L gives an isomorphism of $V^0_{(\Gamma)}(Z)$ with $V^0_{(\Gamma,\gamma)}Z$ and hence of $K^0_\Gamma(Z)$ with $K^0_{(\Gamma,\gamma)}(Z)$. □

End of proof of Theorem 3. To conclude, it is enough to compute $\mathrm{ch}(L)$. Let $\xi \in C^\infty(Z, L)$ be the section $\xi(x) = 1$ for every $x \in Z$. Let ∇ be a Γ-invariant connection on L, one has $\mathrm{ch}(L) = \exp(\omega)$ where $\omega \in H^2(Z/\Gamma, \mathbf{R})$ corresponds to the Γ-invariant 2-form $\theta = \frac{1}{2\pi i} d(\nabla\xi/\xi)$ on Z. Let $\alpha = \frac{1}{2\pi i}\nabla\xi/\xi$, then α is a 1-form on Z, and let us compute for any $g \in \Gamma$ the difference $\alpha - \phi^*\alpha$ where $\phi(x) = xg$ for every $x \in Z$. Since ∇ is Γ-invariant, one has $\phi^*\alpha = \frac{1}{2\pi i}\nabla g(\xi)/g(\xi)$, and as $g(\xi)(x) = e(c(xg, g^{-1}))\,\xi(x)$ one gets $\phi^*\alpha - \alpha = d\psi_g$, where $\psi_g(x) = c(xg, g^{-1})$ for every $x \in Z$. One has $\psi_{g_1g_2} - g_1\psi_{g_2} - \psi_{g_1} = \rho(g_2^{-1}, g_1^{-1})$. This shows that the class of θ in $H^2(Z/\Gamma, \mathbf{R})$ is the pull back of the class of $-\rho$ in $H^2(B\Gamma, \mathbf{R})$, by the classifying map: $Z/\Gamma \to B\Gamma$. □

Using this map $\mathrm{ch}\colon K^*_{(\Gamma,\gamma)}(Z) \to H^*(Z/\Gamma, \mathbf{R})$ we get, by the same five steps as in §6, a map

$$K^*_\gamma(\mathrm{pt}, \Gamma) \xrightarrow{\mathrm{ch}} H_*(B\Gamma, \mathbf{R}).$$

Again as in §6, let ϵ be the map from $B\Gamma$ to a point, and tr_Γ be the canonical trace on $C^*_r(\Gamma, \gamma)$.

THEOREM 6. *For any discrete group Γ and 2-cocycle γ the following diagram is commutative:*

$$\begin{array}{ccc} K^0_\gamma(\mathrm{pt}, \Gamma) & \xrightarrow{\mu} & K_0(C^*_r(\Gamma, \gamma)) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{tr}_\Gamma \\ H_*(B\Gamma, \mathbf{R}) & \xrightarrow{\epsilon^*} & \mathbf{C}. \end{array}$$

The proof is a simple adaptation of the heat equation method to compute the Γ-index of the (Γ, γ)-Dirac operator on a Γ-manifold Z.

Corollary 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of* $\mathbf{R}$, $\Delta = \mathrm{tr}_\Gamma(K_0(C_r^*(\Gamma,\gamma)))$ *contains the group:*

$$\langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho) \rangle .$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho) \rangle$. Thus using Theorem 3 we get:

Corollary 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma,\mathbf{R})/H^2(\Gamma,\mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C_r^*(\Gamma,\gamma))$ by the canonical trace $\zeta = \mathrm{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma,\gamma)$ (for any value of γ), one gets that the corresponding C^*-algebras are isomorphic only when the Γ's are the same (using K_1) and when the γ's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. Foliations

Let V be a C^∞-manifold, and let F be a C^∞-foliation of V. Thus F is a C^∞-integrable sub-vector bundle of TV. As in [33] let G be the holonomy groupoid (graph) of (V,F). The manifold V is assumed to be Hausdorff and second countable. G, however, is a C^∞-manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞-paths

$$\gamma \colon [0,1] \to V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0,1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \underset{r}{\overset{s}{\rightrightarrows}} V$.

Let Z be a possibly non-Hausdorff C^∞-manifold. Assume given a C^∞-map $\rho \colon Z \to V$, set

$$Z \circ G = \{(z,\gamma) \in Z \times G \mid \rho(z) = s(\gamma)\} .$$

A C^∞ *right action of G on Z* is a C^∞-map