2. Orderings of mapping class groups using hyperbolic geometry

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(1) Neville Smythe [23] used the orderability of surface groups to prove that any null-homotopic curve on a surface S is the image under projection of an embedded unknotted loop in $S \times I$.

(2) As pointed out by N. Smythe [16] in response to a question of L. Neuwirth [15, Question N], knot groups are in general not bi-orderable. For instance the trefoil knot group (which is isomorphic to the braid group on three strings B_3), is not bi-orderable. To show this, recall that B_3 contains an element Δ (the "half twist") which is not in the centre, but whose square Δ^2 is. Assume that $>$ is a bi-ordering of B_3, and let $b \in B_3$ be such that $b\Delta \neq \Delta b$, say $b\Delta > \Delta b$. Multiplying this inequality on the left by Δ and on the right by Δ^{-1} would yield $\Delta b > \Delta^2 b \Delta^{-1} = b\Delta^2\Delta^{-1} = b\Delta$, which is a contradiction.

Neuwirth reformulated the question as 'Are knot groups left orderable ?'. A positive answer to this question follows from an observation by J. Howie and H. Short [12] that knot groups are locally indicable (every non-trivial finitely generated subgroup has $\mathbf{Z}$ as a homomorphic image), together with a theorem of Burns and Hale [4] that locally indicable groups are left orderable. The converse of Burns and Hale's theorem is known to be false – see [1] and [9, Theorem 5.3].

(3) We have just seen that B_3 (and hence B_n for all n) is not bi-orderable. Kim and Rolfsen [13] have recently proved that the finite index subgroup PB_n of *pure* braids *is* bi-orderable. However, no bi-ordering of PB_n extends to a left ordering of B_n [20].

(4) The Zero Divisor Conjecture, often attributed to Kaplansky, asserts that if R is a ring without zero divisors and G is a torsion-free group then the group ring RG has no zero divisors. The hypothesis that G be torsion-free is necessary, for if G contains an element x of order n then $(1-x)(1+x+\cdots+x^{n-1}) = 0$ in RG. The conjecture is known to hold for left orderable groups. In fact, it is not hard to see that left orderable groups have the "two unique product" property which implies that the conjecture holds for them (see e.g. [18], and also Delzant [7] and Bowditch [3] for some recent remarks about this property).

2. Orderings of mapping class groups using hyperbolic geometry

In this section we present the construction of orders on mapping class groups of surfaces which we learned from W. P. Thurston, and prove that they all extend the subword-ordering of Elrifai-Morton. The idea comes from the following classical situation as developed by Nielsen. As is well known, every closed surface of genus $g \geqslant 2$ can carry a hyperbolic structure; i.e. there is a homeomorphism between the universal cover $S^{\sim}$ of S and the hyperbolic plane $\mathbf{H}^2$ such that the covering transformations are isometries of $\mathbf{H}^2$. There is a natural closure $S^{\overline{\sim}} \cong \overline{\mathbf{H}}^2$ of $S^{\sim} \cong \mathbf{H}^2$, defined by adding the so-called *circle at infinity* $S^1_\infty = \partial\overline{\mathbf{H}}^2$. Points of this circle can be defined as classes of geodesics, or quasi-geodesics, $\gamma\colon [0,\infty) \to \mathbf{H}^2$, staying a bounded distance apart. The covering action of $\pi_1(S)$ on $S^{\sim}$ extends to an action on $S^{\overline{\sim}}$. So in particular, we have an action of $\pi_1(S)$ on the circle at infinity by homeomorphisms; this action has been much studied (for a good modern exposition of this see [10]). Even stronger, every homeomorphism of the surface lifts and extends to a homeomorphism of $S^{\overline{\sim}}$; however, there is a $\pi_1(S)$-family of possible choices of lift, and therefore we get no well-defined action of $\mathcal{MCG}(S)$ on S^1_∞.

Instead of closed surfaces, Thurston considers surfaces S with nonempty boundary, a finite number of punctures, and $\chi(S) < 0$. Again, one can obtain a hyperbolic structure on S in which ∂S is a geodesic and the punctures are cusps; this time, $S^{\sim}$ is identified with a proper subset of $\mathbf{H}^2$. The boundary of this subset is just the union of the lifts of ∂S; in particular it is a union of geodesics in $\mathbf{H}^2$, and it follows that $S^{\sim}$ is convex in the hyperbolic metric. Moreover, the set of limit points of $S^{\sim}$ on the circle at infinity $\partial\overline{\mathbf{H}}^2$ is a Cantor set in $\partial\overline{\mathbf{H}}^2$. The closure $S^{\overline{\sim}}$ of $S^{\sim}$ in $\overline{\mathbf{H}}^2$, i.e. $S^{\sim}$ with its limit points on the circle at infinity attached, is homeomorphic to a closed disk; $\partial S^{\overline{\sim}}$ is a circle, also containing $S^{\overline{\sim}} \cap \partial\overline{\mathbf{H}}^2$ as a Cantor set.

We now fix, once and for all, a basepoint of $S^{\sim}$ anywhere on $\partial S^{\sim}$. We denote the component of $\partial S^{\sim}$ which contains the base point by Π (see Figure 1). The basepoint projects to a basepoint of S in ∂S, and Π is an infinite cyclic cover of one component of ∂S. We consider the set of geodesics in $S^{\overline{\sim}}$ starting at the basepoint – they are parametrized by the interval $(0,\pi)$, according to their angle with Π. We shall denote by $\widetilde{\gamma}_\alpha$ the geodesic with angle $\alpha \in (0,\pi)$ and by γ_α its projection to S. Since $S^{\sim}$ is hyperbolically convex, each point of $\partial S^{\overline{\sim}}$ can be connected to the basepoint by a unique geodesic (possibly of infinite length) in $S^{\overline{\sim}}$, and for points in $S^{\overline{\sim}} \backslash \Pi$ this is one of the geodesics $\widetilde{\gamma}_\alpha$ with $\alpha \in (0,\pi)$. This construction proves

LEMMA 2.1. *There is a natural homeomorphism between $\partial S^{\overline{\sim}} \backslash \Pi$ and* $(0,\pi)$. □

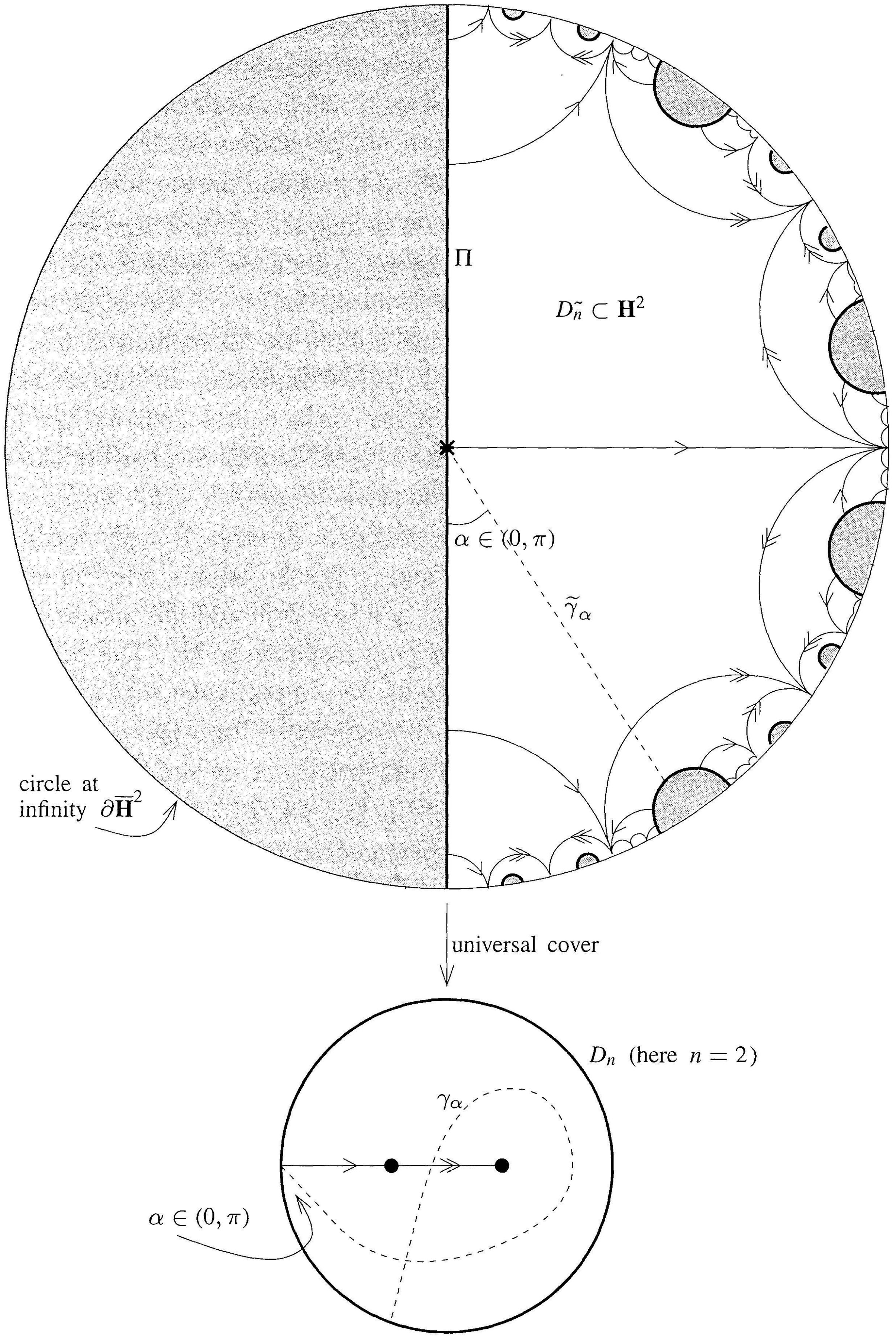

FIGURE 1

Picture of $S^{\sim}$ in $\mathbf{H}^2$ (here S is a twice-punctured disk)

As in the case of closed surfaces, we have an action of $\pi_1(S)$ on $S^{\bar{\sim}}$, which restricts to an action on $\partial S^{\bar{\sim}}$. However, this time we have more:

PROPOSITION 2.2. *There is a natural action by orientation preserving homeomorphisms of* $\mathcal{MCG}(S)$ *on* $\partial S^{\bar{\sim}} \backslash \Pi \cong (0, \pi)$.

Proof. Every homeomorphism $\varphi \colon S \to S$ has a canonical lift $\widetilde{\varphi} \colon S^{\sim} \to S^{\sim}$, namely the one that fixes the basepoint of $S^{\sim}$, and thus all of Π. Moreover, $\widetilde{\varphi}$ has an extension $\bar{\widetilde{\varphi}} \colon S^{\bar{\sim}} \to S^{\bar{\sim}}$. The restriction of this homeomorphism to $\partial S^{\bar{\sim}}$ is invariant under isotopy of φ, and fixes Π, and thus yields a well-defined orientation-preserving homeomorphism of $\partial S^{\bar{\sim}} \backslash \Pi$. (Note that there is no requirement for S to be orientable here.) $\square$

COROLLARY 2.3. $\mathcal{MCG}(S)$ *is left orderable.*

Proof. No nontrivial element of $\mathcal{MCG}(S)$ acts trivially on $(0, \pi)$, because if such an element existed, it would in particular fix all liftings of the basepoint of S, and thus induce the identity-homorphism on $\pi_1(S)$; by [2, Corollary 1.8.3] it would then be isotopic to the identity, in contradiction with the hypothesis. The result now follows from Remark 1.2(2), because $(0, \pi)$ is homeomorphic to $\mathbf{R}$.

However, there is an elementary proof in our situation. We choose arbitrarily a finite generating set of $\pi_1(S)$, and denote the end points of the liftings of these elements by $s_1, \ldots, s_k \in (0, \pi)$. A left order on $\mathcal{MCG}(S)$ is now defined inductively: if $\varphi(s_1) > s_1$ then $\varphi > 1$ (and the same with $>$ replaced by $<$); if $\varphi(s_1) = s_1$, but $\varphi(s_2) > s_2$, then $\varphi > 1$ as well, and so on; this is a total order, because we have that $\varphi(s_i) = s_i$ for all i if and only if $\varphi = 1$. $\square$

However, for the rest of the paper we shall be less interested in orderings of this type, but rather in orderings induced by the orbits of single geodesics, i.e. in orderings of the type introduced in Remark 1.2(1).

We recall the definition of a *positive Dehn twist* along a simple closed curve τ in the surface S: it can be characterised as a homeomorphism $S \to S$ which maps all but an annular neighbourhood of τ identically, and sends any arc that crosses τ to an arc that, upon entering the annular neighbourhood, turns left, spirals exactly once along τ, and then turns right to leave the annular neighbourhood through its other boundary component and continue as before. For example in the case of a punctured disk, if $\Delta \in B_n$ denotes the "half-twist braid", then Δ^2 is a Dehn twist along a curve parallel to the boundary of the disk.

PROPOSITION 2.4. *For the positive Dehn twist T along any simple closed geodesic τ in S we have $T(\alpha) \geqslant \alpha$ for any $\alpha \in (0, \pi)$. If γ_α intersects τ at least once, then the inequality is strict.*

Proof. If γ_α is disjoint from τ, then $T(\alpha) = \alpha$. If, on the contrary, γ_α intersects τ, and hence any curve isotopic to τ, any number of times (possibly infinitely often), then we denote by $T_i(\gamma_\alpha)$ $(i \in \mathbf{N})$ the curve obtained from γ_α by applying the Dehn twist to the first i intersections of γ_α with τ and ignoring all following intersections; we denote by $T_i(\alpha)$ its end point in $\partial \tilde{\bar{D}}_n \backslash \Pi$. We have $T(\alpha) = \lim_{i \to \infty} T_i(\alpha)$.

We now claim that $(T_i(\alpha))_{i \in \mathbf{N}}$ is a strictly increasing sequence. To simplify notation, we shall prove the special case $T_1(\alpha) > \alpha$, the proof in the general case is exactly the same. In the universal cover $\tilde{\bar{D}}_n$ we consider the lifting of the curve $T_1(\gamma_\alpha)$: starting at the basepoint, it sets off along $\tilde{\gamma}_\alpha$, up to the first intersection with some lifting $\tilde{\tau}$ of τ. There it turns left, walks along $\tilde{\tau}$ up to the next preimage of the intersection point, where it encounters a different lifting $\tilde{\gamma}'_\alpha$ of γ_α. There it turns right, following this lifting all the way to $\partial \tilde{\bar{D}}_n \backslash \Pi$. The crucial point now is that $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}'_\alpha$ intersect $\tilde{\tau}$ at the same angle, because the two intersections are just different liftings of the same intersection between γ_α and τ in D_n. It follows that $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}'_\alpha$ do not intersect, not even at infinity, for if they did they would determine a hyperbolic triangle in $\tilde{\bar{D}}_n$ two of whose interior angles already add up to 180 degrees, which is impossible. This implies the claim, and thus proves the proposition. $\square$

COROLLARY 2.5. *All total orderings of the braid group B_n considered in this paper extend the subword-ordering of Elrifai-Morton* [8, 25]. *More precisely, if a curve τ in D_n encloses a precisely twice punctured disk and $T^{1/2}$ is the positive half-Dehn twist along τ interchanging the two punctures then $T \circ \varphi > \varphi$ for any $\varphi \in B_n$ and any ordering $>$ of Thurston-type.*

Proof. It suffices to prove that $T^{1/2}(\alpha) \geqslant \alpha$ for all $\alpha \in (0, \pi)$. If there existed an $\alpha \in (0, \pi)$ with $T^{1/2}(\alpha) < \alpha$ then it would follow that $T(\alpha) = T^{1/2} \circ T^{1/2}(\alpha) < T^{1/2}(\alpha) < \alpha$ (where the first inequality holds since $T^{1/2}$ is orientation preserving), in contradiction with the proposition. $\square$

REMARK 2.6. Here is an example of an ordering $\prec$ of B_n that does not arise from Thurston's construction: if "$<$" is any ordering of Thurston-type, then we define an element $\varphi \in B_n$ to be in the positive cone of $\prec$ if either $ab(\varphi)$ is positive, where $ab\colon B_n \to \mathbf{Z}$ is the abelianization, or if $ab(\varphi) = 0$

and $1_{B_n} < \varphi$. In this ordering the commutator subgroup is convex [19], and we leave it to the reader to verify that no Thurston-type ordering has this property.

3. Main results

We shall mainly be interested in the case $S = D_n$ $(n \geq 2)$, where D_n is the closed unit disk in $\mathbf{C}$, with n punctures lined up in the real interval $(-1, 1)$; in this case the mapping class group is a braid group: $\mathcal{MCG}(D_n) = B_n$. We recall that for $\alpha \in (0, \pi)$ we denote by γ_α the geodesic which starts at the basepoint with angle α with ∂S, and by $\widetilde{\gamma}_\alpha$ its preimage in the universal cover starting at the basepoint of $S^\sim$.

DEFINITION 3.1. A geodesic γ_α, $\alpha \in (0, \pi)$, is said to be of *finite type* if it satisfies at least one of the following conditions:

(a) there exists a finite initial segment γ_α^t such that any two punctures that lie in the same path component of $S \backslash \gamma_\alpha^t$ also lie in the same path component of $S \backslash \gamma_\alpha$, or

(b) it falls into a puncture, or

(c) it spirals towards a simple closed geodesic.

If a geodesic γ_α is not of finite type then we say it is of *infinite type*. We also define the ordering of $\mathcal{MCG}(S)$ induced by a geodesic γ_α to be of finite or infinite type if γ_α is of finite or infinite type.

An infinite type geodesic looks as follows. All its self intersections occur in some finite initial segment γ_α^t. At least one of the path components of $S \backslash \gamma_\alpha^t$ contains three or more punctures in its interior, and the closure of $\gamma_\alpha \backslash \gamma_\alpha^t$ is a geodesic lamination without closed leaves inside such a component. In particular, there is a pair of punctures which are separated by the whole geodesic, but not by any finite initial segment. (Note that the geodesic $\gamma_\alpha \backslash \gamma_\alpha^t$ is isolated from both sides – in this it is very different from leaves of geodesic laminations on surfaces without boundary.)

DEFINITION 3.2. For a geodesic γ_α of finite respectively infinite type we say that it *fills the surface in finite* respectively *infinite time* if all punctures lie in different path components of $S \backslash \gamma_\alpha$. By contrast, a geodesic γ_α *does not fill* the surface if $S \backslash \gamma_\alpha$ has a path component that contains two punctures.