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starts at the tip of D_{cc} (i.e. at the same point as $\Gamma_i \cap D_c$ and $\varphi(\Gamma_i) \cap D_c$), and falls into one of the punctures in the right half of D_{cc}. By construction, $\gamma_\alpha \cap D_{cc}$ is reduced with respect to σ, since both are geodesics, and the first component of $\varphi(\gamma_\alpha) \cap D_{cc}$ is even disjoint from σ. In the universal cover we now have that the lifting $\widetilde{\sigma}$ of σ ends on the circle at infinity, thus separating $\widetilde{D}_{cc}$ into two components, the left one containing the lift of $\varphi(\gamma_\alpha) \cap D_{cc}$, and the right one the lift of $\gamma_\alpha \cap D_{cc}$. Thus lifts of these two curves, not being allowed to intersect any component of $\partial D_{cc}^{\sim}$ and $\partial D_c^{\sim}$ more than once, go on to hit different points of $\partial D_n^{\sim}$, with $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ staying more to the left than $\widetilde{\gamma}_\alpha$. This completes the proof of the third case, and thus of Theorem 6.1. □

Proof of Theorem 3.3 (b). If γ_α fills D_n, then $C(\gamma_\alpha)$ is a total curve diagram, and thus induces a *total* ordering of B_n. By Corollary 6.2, the ordering of B_n associated to the point $\alpha \in (0, \pi)$ agrees with this ordering. □

Proof of Theorem 3.4 (b). For any two geodesics γ_α and γ_β of finite type one can work out their associated curve diagrams $C(\gamma_\alpha)$ and $C(\gamma_\beta)$. By Corollary 6.2 it is sufficient to decide whether or not the orderings associated to the two curve diagrams coincide, which can be done by Theorem 5.2. □

Proof of Theorem 3.5. It only remains to be proved that $N_n = M_n$ (where M_n is given in Proposition 5.3), i.e. that every curve diagram is realized up to loose isotopy as $C(\gamma_\alpha)$ for some geodesic γ_α, $\alpha \in (0, \pi)$. This is left as an exercise to the reader. □

7. Orderings associated to geodesics of infinite type

In this section we prove the results concerning orderings of infinite type, and explain the essential differences between finite and infinite type orderings.

We start by describing in more detail than in Section 3 the structure of geodesics of infinite type. We define the *curve diagram* $C(\gamma_\alpha)$ *associated to a geodesic of infinite type* by precisely the same inductive construction procedure as in the finite type case. Except for a finite initial segment, the last arc Γ_j will lie in some path component D_c of $D_n \backslash N\Gamma_{0\cup\cdots\cup j-1}$, the only difference with the finite type case is that Γ_j goes on for ever, without falling into a puncture and without spiralling. The closure of Γ_j inside this critical component D_c is a geodesic lamination; the lamination has no closed leaves, for such a leaf would have to be the limit of an infinite spiral of Γ_j (see [17, Appendix]). All self-intersections of the geodesic γ_α occur inside the finite initial segment up to the entry into the punctured disk D_c ; in particular, there are only finitely many self-intersections.

Proof of Theorem 3.3 (c). We are studying the set

$$\mathcal{I} := \{\alpha \in (0, \pi) \mid \gamma_\alpha \text{ is of infinite type}\}.$$

The proof uses standard results from the theory of geodesic laminations and the Nielsen-Thurston classification of surface automorphisms [5, 17].

That $\mathcal{I}$ has uncountably many elements follows from the fact that there are uncountably many geodesic laminations of D_n, only countably many of which fall into infinite spirals. A more practical way of seeing this is to choose arbitrarily a fundamental domain of D_n by fixing n geodesic arcs, e.g. as shown in Figure 1. Thus the fundamental domain is a $2n+1$-gon with one boundary edge corresponding to ∂D_n and n pairs of boundary edges which are identified in D_n. A segment of the geodesic between any two sucessive intersections with the boundary of the fundamental domain consists of an embedded arc connecting different edges of the $2n+1$-gon. Hence constructing a geodesic of infinite type amounts to choosing an infinite "cutting sequence" of the geodesic with the boundary arcs of the fundamental domain. Often the choice will be forced upon us by the requirement that the geodesic be embedded, but there will be an infinite number of times when we have a genuine choice. Thus the set of all possible sequences of choices is uncountable.

The cutting sequence approach also makes it clear why any neighbourhood of an $\alpha \in \mathcal{I}$ in $(0, \pi)$ contains points $\alpha' \neq \alpha$ of $\mathcal{I}$ as well as $\beta \in (0, \pi) \setminus \mathcal{I}$. Given $\alpha \in (0, \pi)$ and $\epsilon > 0$, there exists an $N_\epsilon \in \mathbf{N}$ such that all geodesics γ_δ whose cutting sequences agree with the one of γ_α for at least N_ϵ terms satisfy $|\alpha - \delta| < \epsilon$. Now for any $\alpha \in \mathcal{I}$ and $\epsilon > 0$ we can find a geodesic $\gamma_{\alpha'}$ of infinite type whose cutting sequence diverges from the one of γ_α only after the $N_\epsilon{}^{\text{th}}$ term. On the other hand, we can construct a geodesic γ_β with $|\alpha - \beta| < \epsilon$ which fills D_n in finite time: just choose it to have a cutting sequence which agrees with the one of γ_α for N_ϵ terms, and to then career off along some path which decomposes D_n into disks and once-punctured disks.

Finally, the last part of Theorem 3.3(c) holds because each of the countably many elements of B_n fixes only a countable number of points $\alpha \in (0, \pi)$ with the property that γ_α fills D_n. In order to see this, we note that for *irreducible* elements of B_n Theorem 5.5 of [5] states that there is only a finite number of fixed points on the circle at infinity. If an element φ of B_n is *reducible*, then we leave it to the reader to check that the result follows from the following facts:

(1) One can find a maximal invariant system C of disjoint properly embedded arcs and circles in D_n.

(2) If φ acts nontrivially on a component of $D_n \backslash C$ which is cut in a nontrivial way by a *finite* segment of γ_α, then it acts nontrivially on γ_α (for if it didn't then the collection C would not be maximal).

(3) A geodesic γ_α that fills D_n has to enter every component of $D_n \backslash C$ at least once, and φ acts nontrivially either on the first or, failing that, on the second component of $\gamma_\alpha \cap (D_n \backslash C)$ (because it cannot act trivially on two adjacent components of $D_n \backslash C$).

(4) There is a countable infinity of isotopy classes of embedded arcs from the basepoint of D_n to C. □

We recall from the beginning of the section that to every geodesic γ_α of infinite type we have associated a "critical disk" D_c which contains most of the last arc of $C(\gamma_\alpha)$. The fundamental property of geodesics of infinite type which we shall use several times is the following.

LEMMA 7.1. *For any geodesic of infinite type γ_α and for any $\epsilon > 0$ there exists a geodesic γ_{α^+} with $\alpha^+ \in (\alpha, \alpha + \epsilon)$ such that γ_{α^+} falls into a puncture and has no self-intersections inside D_c.*

Proof. It suffices to prove the lemma in the special case $D_c = D_n$, i.e. when the geodesic γ_α is embedded. We suppose, for a contradiction, that there exists an $\epsilon > 0$ such that no γ_β with $\beta \in (\alpha, \alpha + \epsilon)$ is embedded *and* falls into a puncture. Our aim is to reach the contradiction that γ_α ends in an infinite spiral.

We continue to use the notions concerning cutting sequences introduced above: we choose arbitrarily a fundamental domain, and we shall denote by γ_α^k the initial segment of γ_α up to its k^{th} intersection with the boundary of the fundamental domain. We recall that, given γ_α and $\epsilon > 0$, we can find an $N = N_\epsilon \in \mathbf{N}$ such that any geodesic γ_β with $\gamma_\beta^N = \gamma_\alpha^N$ satisfies $|\alpha - \beta| < \epsilon$. We now consider the arc γ_α^{N+1} : it ends on some boundary arc of the fundamental domain which we denote a. The orientation of γ_α gives rise to a notion of the part of a "to the left" and "to the right of" the end point of γ_α^{N+1}. The arc γ_α^{N+1} has an intersection with the interior of the "left" part of a, for if this were not the case we could obtain an embedded arc γ_β

with $\beta \in (\alpha, \alpha+\epsilon)$ by adjoining to the end point of γ_α^N an arc falling into the puncture at the left end of a; this would contradict the hypothesis. Thus it makes sense to define $\Gamma \subseteq D_n$ to be the union of γ_α^{N+1} and a segment of a from the end point of γ_α^{N+1} to the left, up to the next intersection with γ_α^{N+1} (see Figure 10).

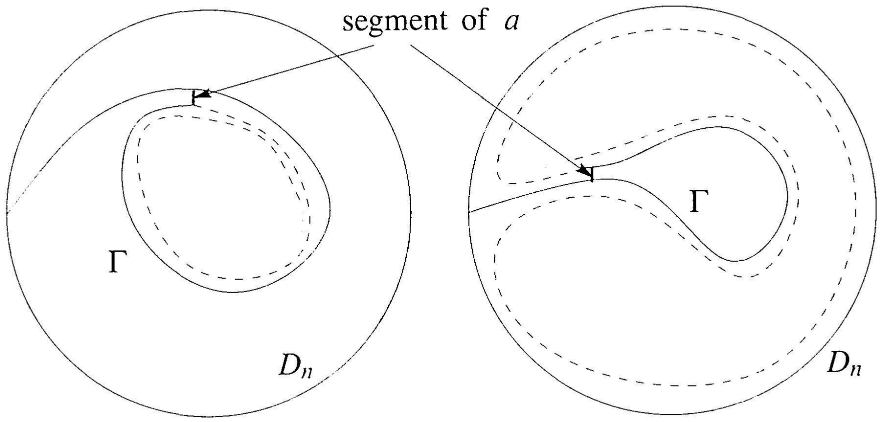

FIGURE 10

The two possible shapes of Γ, and (dashed) the resulting geodesic γ_α

We now observe that $D_n \backslash \Gamma$ has two path components, each containing at least one puncture; moreover, γ_α cannot intersect any geodesic arc connecting two punctures in the same component, because the first time it did we could drop it into the puncture at the left end of the arc and obtain a contradiction as before. It follows that γ_α has to spiral along the boundary of one of the components of $D_n \backslash \Gamma$. $\square$

PROPOSITION 7.2. *All orderings, even partial ones, arising from geodesics* γ_α *of infinite type are non-discrete.*

Proof. We shall prove the following stronger statement: for any $\epsilon > 0$ there exists an element $\varphi \in \mathcal{MCG}(D_n) = B_n$ such that $\varphi(\alpha) \in (\alpha, \alpha+\epsilon)$.

We choose α^+ as in the previous lemma. We consider the boundary curve τ of a regular neighbourhood of $\partial D_c \cup \gamma_{\alpha^+}$ in D_c. This curve τ is disjoint from γ_{α^+}, while any curve isotopic to τ necessarily intersects γ_α. Thus for the positive Dehn twist T along τ we have that $T(\alpha) > \alpha$ (by Proposition 2.4), and that $T(\alpha^+) = \alpha^+$. It follows that $T(\alpha) \in (\alpha, \alpha^+) \subseteq (\alpha, \alpha+\epsilon)$. $\square$

Proof of Theorem 3.4 (a). Given a geodesic γ_α of finite, and a geodesic γ_β of infinite type, our aim is to prove that γ_α and γ_β cannot induce the same orderings of B_n.

As seen in Corollary 6.2, orderings arising from geodesics which fill the surface in finite time are the same as orderings arising from total curve diagrams, which are discrete by Lemma 4.5. By contrast, we have from Proposition 7.2 that infinite type orderings are not discrete. This proves the theorem in the special case where the finite type geodesic fills the surface.

In the case where the finite type geodesic γ_α does *not* fill the surface, we consider the subsurface $D_\alpha := D_n \backslash NC(\gamma_\alpha)$, i.e. the maximal subsurface with geodesic boundary which is disjoint from γ_α. We observe that D_α is a disjoint union of disks, each containing at least two punctures. Any homeomorphism φ of D_n with support in D_α has the property that $\varphi(\alpha) = \alpha$.

If $D_\alpha \cap \gamma_\beta \neq \varnothing$ then there exists a homeomorphism φ with support in D_α such that $\varphi(\beta) \neq \beta$, and it follows that the orderings induced by α and β are different.

If, on the other hand, $D_\alpha \cap \gamma_\beta = \varnothing$, then we squash each component of D_α to a puncture; the result is a disk with say m punctures, where $m < n$, which we denote D_m. We now consider the subgroup B_m^P of $B_m = \mathcal{MCG}(D_m)$ of all mapping classes which fix those punctures of D_m that came from squashed components of D_α. This is a finite index subgroup of B_m, and the orderings of B_n determined by α and β induce quotient orderings on B_m^P. Another way to describe these quotient orderings is to repeat the Thurston-construction for the disk D_m: one can equip D_m with a hyperbolic metric, and then the geodesics γ_α and γ_β project to quasigeodesics in D_m. These quasigeodesics determine points at infinity of the universal cover of D_m, and hence give rise to orderings of B_m.

The geodesic in D_m which is homotopic to the projection of γ_α is again of finite type; the crucial observation now is that it fills D_m, so that the quotient ordering on B_m^P is discrete by Lemma 4.5. Similarly, a geodesic in D_m homotopic to the projection of γ_β is again of infinite type, hence induces, by Proposition 7.2 a non-discrete ordering on B_m, and thus also on the finite-index subgroup B_m^P. So the α- and β-orderings on B_n give rise to different quotient orderings on B_m^P, and are therefore different. □

As seen above, every geodesic of infinite type gives rise to a curve diagram "of infinite type", which is like a curve diagram of finite type, except that the arc with maximal label is, up to isotopy, an infinite geodesic which does not fall into a puncture or a spiral. All but a finite initial segment of this arc lies in the "critical disk" D_c. There is an obvious generalisation of the notion of loose isotopy:

DEFINITION 7.3. Two curve diagrams of infinite type are *loosely isotopic* if they are related by (1) continuous deformation, i.e. a path in the space of all curve diagrams of infinite type; and (2) pulling loops around punctures tight.

This is exactly the same as in the finite type case, except that no "pulling loops around punctures tight"-procedure is defined for the last arc. We are now ready to state and prove the main classification theorem for orderings of B_n of infinite type.

THEOREM 7.4. *Two geodesics γ_α and γ_β of infinite type give rise to the same (possibly partial) ordering of B_n if and only if their associated curve diagrams $C(\gamma_\alpha)$ and $C(\gamma_\beta)$ are loosely isotopic.*

Proof. By the results in the previous sections, it suffices to prove that two *embedded* geodesics γ_α and γ_β of infinite type give rise to the same ordering of B_n if and only if $\beta = \Delta^{2k}(\alpha)$ for some $k \in \mathbf{Z}$, i.e. if γ_α and γ_β are related by a slide of the starting point around ∂D_n.

The implication "$\Leftarrow$" is clear. Conversely, for the implication "$\Rightarrow$", we suppose that γ_α and γ_β are not related by a slide of the starting point, and without loss of generality we say $\alpha > \beta$. Our aim is to construct a homeomorphism which is positive in the α- and negative in the β-ordering, i.e. which sends α "more to the left" and β "more to the right". Our argument will be a refinement of the proof of the implication "$\Rightarrow$" of 5.2(a).

By Lemma 7.1 we can construct embedded geodesics $\gamma_{\alpha+}$ and $\gamma_{\beta+}$ which fall into punctures, and lie an arbitrarily small amount to the left of γ_α respectively γ_β. We define the curves $\tau_{\alpha+}$ and $\tau_{\beta+}$ to be the geodesic representatives of the boundary curves of regular neighbourhoods in D_n of $\partial D_n \cup \gamma_{\alpha+}$ and $\partial D_n \cup \gamma_{\beta+}$ respectively. We denote by $T_{\alpha+}$ respectively $T_{\beta+}$ the positive Dehn twists along these curves. Our desired homeomorphism will be of the form $T_{\alpha+}^{-k} \circ T_{\beta+}$, with carefully chosen values of α^+ and β^+, and $k \in \mathbf{N}$ very large.

We also define the two-sided infinite geodesic τ_α to be the geodesic which is disjoint from γ_α, and isotopic to the boundary of a neighbourhood of $\gamma_\alpha \cup \partial D_n$ in D_n. More formally, in the universal cover $\widetilde{D_n}$ we consider two liftings of γ_α, namely $\widetilde{\gamma_\alpha}$ (which starts at the basepoint of $\widetilde{D_n}$), and the lifting whose starting point also lies on Π and is obtained from the basepoint of $\widetilde{D}_n$ by lifting the path once around ∂D_n. The end points of these geodesics lie on the circle at infinity, and τ_α is just the projection of the geodesic connecting them.

Since γ_α and γ_β are not loosely isotopic, we have that γ_β intersects τ_α. By choosing β^+ sufficiently close to β we can now achieve that the initial segments of γ_β and γ_{β^+} up to their first point of intersection with τ_α are isotopic with end points sliding in τ_α. This gives our choice of β^+, and it remains to choose α^+ and k.

The crucial observation concerning τ_α is that it can be arbitrarily closely approximated by the curves τ_{α^+}, by choosing α^+ sufficiently close to α. More precisely, in the universal cover $\tilde{D_n}$ we consider the preimages of τ_α and of τ_{α^+}. Each of them has infinitely many path components; we choose one distinguished component for each, namely the first ones that γ_β intersects. Our observation now is that as α^+ tends to α, the end points of the distinguished component of the preimage of τ_{α^+} tend to the end points of the distinguished component of the preimage of τ_α.

We now turn to the choice of α^+. By Proposition 2.4 we have that $T_{\beta^+}(\alpha) > \alpha$. By Lemma 7.1 we can now choose α^+ close to α such that $T_{\beta^+}(\alpha) > \alpha^+ > \alpha$. By possibly pushing α^+ even closer to α, we can in addition insist (by the observation concerning τ_α above) that the initial segments of γ_β and γ_{β^+} up to their first point of intersection with τ_{α^+} are also isotopic with end points sliding in τ_{α^+}. This gives our choice of α^+.

We have arrived at the following setup: we have the three points $\beta^+ = T_{\beta^+}(\beta^+) > T_{\beta^+}(\beta) > \beta$ in $\partial\tilde{D_n}\backslash\Pi$, and they all lie between the two end points δ_l and δ_r of the distinguished lifting of τ_{α^+} (here the indices l and r stand for "left" and "right", so $\delta_l > \delta_r$). For any point δ with $\delta_l > \delta > \delta_r$ we consider the action of the positive Dehn twist T_{α^+} on the geodesic γ_δ. We observe that the limit $\lim_{k\to\infty} T_{\alpha^+}^{-k}(\delta) = \delta_r$. In particular for $\delta := \beta^+$ it follows that for sufficiently large k we have $T_{\alpha^+}^{-k}(\beta^+) < \beta$. This gives our choice of k.

To summarise, we have

$$T_{\alpha^+}^{-k} \circ T_{\beta^+}(\alpha) > T_{\alpha^+}^{-k}(\alpha^+) = \alpha^+ > \alpha$$

and

$$T_{\alpha^+}^{-k} \circ T_{\beta^+}(\beta) < T_{\alpha^+}^{-k} \circ T_{\beta^+}(\beta^+) = T_{\alpha^+}^{-k}(\beta^+) < \beta,$$

i.e. $T_{\alpha^+}^{-k} \circ T_{\beta^+}$ is positive in the α-, but negative in the β-ordering. □

Proof of Theorem 3.4 (c). This is an immediate consequence of Theorem 7.4. □

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