Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY
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Kapitel: §2. Notations and some known facts
DOI: https://doi.org/10.5169/seals-64804

(Thm. 1) by a very simple general argument that the operator $\mathcal{F}_p$ in (3) is surjective only in the following obvious cases: (i) $p = p' = 2$ or (ii) G finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group G whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary L^p-spaces (stated in § 2). From this we deduce fairly simply that for any infinite locally compact commutative group G, the inequality (1) cannot be extended to the case $2 < p < \infty$; the exact statement is given as Thm. 2 in § 3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in § 2, state and prove the facts alluded to above in § 3 and add a few historical comments in § 4; a short appendix (§ 5) is added to explain the L^p-isomorphism theorem stated in § 2.

We have not tried to extend our theorems to the case of G non-commutative, using for $\widehat{G}$ the set of all equivalence classes of continuous unitary irreducible representations of G. For G compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in § 4.

§ 2. Notations and some known facts

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple (X, Σ, μ) where Σ is a σ-algebra of subsets of the abstract set X and $\mu \colon \Sigma \to [0, \infty]$ is a σ-additive positive measure; no finiteness or σ-finiteness conditions will be imposed a priori on μ. Then $L^p(\mu)$, $1 \leq p \leq \infty$, will denote the usual Banach space associated with Σ-measurable complex-valued functions f defined on X with $\|f\|_p < \infty$, $\|f\|_p$ being the standard L^p-norm with respect to μ. If G is a locally compact commutative group (always supposed to be Hausdorff), $L^p(G)$, $1 \leq p \leq \infty$, will stand for the associated L^p-space obtained by fixing some Haar (invariant) measure on G, and $\widehat{G}$ will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma \colon G \to \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}.$$

For a given Haar measure on G, the Haar measure on $\widehat{G}$ will always be fixed in such a way that the Plancherel formula be valid in $L^2(G)$; if $f \in L^1(G)$, $\widehat{f}$ will be defined by (2) above.

Recall that for any measure space (X, Σ, μ), the dual Banach space of $L^p(\mu)$ is $L^{p'}(\mu)$ whenever $1 < p < \infty$, i.e. in symbols

$$(L^p(\mu))' = L^{p'}(\mu) \tag{4}$$

whether μ is σ-finite or not. Here and elsewhere,

$$p' = \frac{p}{p-1}, \qquad 1 < p < \infty,$$

and $1' = \infty$; for (4) to hold for $p = 1$, $p' = \infty$, one needs some conditions on μ (σ-finiteness of μ is sufficient but not necessary). Nevertheless,

$$(L^p(G))' = L^{p'}(G)$$

holds for all $p, 1 \leq p < \infty$, and any locally compact group G. We shall not, however, need this fact.

Two Banach spaces E, F are called isomorphic if there is an isomorphism $u \colon E \to F$ where u is a continuous linear bijection; it is well-known that $u^{-1} \colon F \to E$ is then automatically continuous. The following is proved in [C]: *if (X, Σ, μ), $(Y, \mathcal{J}, \nu)$ are any two measure spaces and $1 \leq p, q \leq \infty$ then $L^p(\mu)$ isomorphic to $L^q(\nu)$ implies necessarily that $p = q$ provided that $L^p(\mu)$ or $L^q(\nu)$ is infinite dimensional.*

We shall refer to this statement as the *L^p-isomorphism theorem*; as indicated in §5, this is an easy consequence of the theory of types and cotypes for Banach spaces. The same reasoning proves (cf. §5) that if (X, Σ, μ) is any measure space such that $L^1(\mu)$ is infinite dimensional and Y is any locally compact Hausdorff space then $L^1(\mu)$ cannot be isomorphic to $C_0(Y)$ (or to $C(Y)$). Here $C(Y)$ is the Banach space of all bounded complex-valued functions on Y endowed with the sup norm, and $C_0(Y)$ is the subspace of those continuous complex-valued functions in Y which vanish at ∞; if Y is compact, we put $C_0(Y) = C(Y)$.

Recall that if $f \in L^1(G)$, G any locally compact commutative group, then $\widehat{f} \in C_0(\widehat{G})$; cf. [HR] vol. 2, p. 212; this fact is sometimes referred to as the Riemann-Lebesgue lemma for G.