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§ 5. Appendix

Here we outline a simple proof of the L^p-isomorphism theorem stated in § 2; the proof uses the notion of type and cotype of Banach spaces and follows [C].

Definition. A Banach space E is of *type* p $(1 \leq p \leq 2)$ if there is a finite positive number C_p such that for all choices of $x_1, \dots, x_n$ in E, $n = 1, 2, \dots$, we have

$$2^{-n} \sum_{\varepsilon_1 \cdots \varepsilon_n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \leq C_p \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p},$$

where $\sum_{\varepsilon_1 \cdots \varepsilon_n}$ stands for the sum of the 2^n quantities obtained by letting each ε_j taking the values $+1$ or -1. E is said to have *exact type* p if it is of type p but not of type $\widetilde{p} > p$.

A Banach space E is of *cotype* q $(2 \leq q \leq \infty)$ if there is a finite positive number c_q such that for all choices of $x_1, \dots, x_n$ in E, $n = 1, 2, \dots$, we have

$$2^{-n} \sum_{\varepsilon_1 \cdots \varepsilon_n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \geq c_q \Big(\sum_{j=1}^{n} \|x_j\|^q \Big)^{1/q}.$$

E is said to have *exact cotype* q if it is of cotype q but not of cotype $\widetilde{q} < q$.

It is obvious that exact type or cotype is an isomorphism invariant. It can be shown that for any measure space (X, Σ, μ) giving rise to infinite dimensional $L^p(\mu)$-spaces we have the following:

- *$L^p(\mu)$ has exact type p if $1 \leq p \leq 2$, exact type 2 if $2 \leq p < \infty$ and exact type 1 if $p = \infty$;*
- *$L^p(\mu)$ has exact cotype 2 if $1 \leq p \leq 2$, exact cotype p if $2 \leq p < \infty$ and exact cotype ∞ if $p = \infty$.*

All this and more is completely proved in [C]; a reference for the general theory of types and cotypes is [DJT].

Suppose now that $L^p(\mu)$ and $L^q(\nu)$ are infinite dimensional and isomorphic where $1 \leq p,\ q \leq \infty$, $(X, \sum, \mu)$, $(Y, \mathcal{J}, \nu)$ being any two measure spaces; we shall prove that $p = q$. Without loss of generality, we may suppose that if $p \neq q$ then $p < q$; this would lead to a contradiction as shown below.

(i) If $1 \leq p < q \leq 2$ then

$$\text{exact type of } L^p(\mu) = p < \text{exact type of } L^q(\nu) = q,$$

which excludes any isomorphism between $L^p(\mu)$, $L^q(\nu)$.

(ii) If $1 \leq p < 2$, $2 \leq q < \infty$ then

$$\text{exact type of } L^p(\mu) = p < \text{exact type of } L^q(\nu) = 2,$$

which excludes any isomorphism between $L^p(\mu)$, $L^q(\nu)$.

(iii) If $2 \leq p < q < \infty$ then $1 < q' < p' \leq 2$; if $L^p(\mu)$, $L^q(\nu)$ were isomorphic then their duals $L^{p'}(\mu), L^{q'}(\nu)$ would be isomorphic, which is impossible in view of (i).

(iv) If $1 < p < \infty$, $q = \infty$ then $L^p(\mu)$ has exact type equal to $\min(p, 2) > 1$ whereas $L^\infty(\nu)$ has exact type 1; thus $L^p(\mu)$ is not isomorphic to $L^\infty(\nu)$ (a fact which is obvious on the grounds of reflexivity as well).

(v) Finally, let $p = 1$, $q = \infty$; then $L^1(\mu)$ is not isomorphic to $L^\infty(\nu)$ since the exact cotype of $L^1(\mu)$ is 2 and the exact cotype of $L^\infty(\nu)$ is ∞.

This completes the proof of the L^p-isomorphism theorem.

A proof that no infinite dimensional $L^1(\mu)$ can be isomorphic to any $C_0(Y)$ or $C(Y)$ (Y any locally compact Hausdorff space) can be based on the same ideas as (v) above. The exact cotype of $L^1(\mu)$ is 2 whereas the exact cotype of any infinite dimensional $C_0(Y)$ or $C(Y)$ is ∞ (exactly as in the case of $L^\infty(\mu)$). This excludes the possibility of any isomorphism between $L^1(\mu)$ and $C_0(Y)$ or $C(Y)$.

REMARK. The L^p-isomorphism theorem seems to be known to various specialists; however, I know of no explicit formulation or proof of it in complete generality except for that in [C].

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